

AN
INTRODUCTION
ARITHMETIC
AND
ALGEBRA;
VOL. II.

By THOMAS MANNING.

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INTRODUCTION

ARITHMETIC

AND

ALGEBRA



BY F. H. C. 12

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IN addition to the general acknowledgements of obligation to former writers, which accompanied the first Volume, nothing, I believe, remains to be stated, except that the method, given in page 21, of determining the answer to questions belonging to the Rule of Three, is taken, with very little alteration, from Dr. Maskelines preface to Taylor's Logarithms.

With respect indeed to many of the subjects treated of in the following pages, a strict adherence to the original plan of suffering no demonstrable proposition to pass without a demonstration, has precluded the author from availing himself of the labors of others so fully and frequently as in his first volume.

Those, who have been accustomed to implicit confidence in rules, will perhaps at first sight condemn

denn as useless the full explanation given in this volume of the application of Algebra to rectilinear Geometry.—Of such the author requests a thorough and impartial examination into the foundation and connexion of their ideas on this subject, in order to ascertain whether they have not been in the habit of substituting analogy for demonstration.

This volume contains simple and easy demonstrations of the logarithmic series, founded on the pure principles of Algebra, which the author considers as possessing many advantages over the fluxional demonstrations, besides that of not requiring the establishment of any new principle. These demonstrations differ essentially from any that he has been able to meet with: but he does not wish to conceal that they were suggested by a theorem in M. Le Grange's most learned and ingenious work, entitled "*Theorie des fonctions Analytiques*" without which they probably would never have appeared in their present simple form.

An anxious desire of keeping the size of this volume within the limits of his engagement, has prevented the author from pursuing some points
to

to that extent, his inclinations might otherwise have led him—at the same time he chose rather to proceed as far as those limits permitted, than omit any thing, which he thought might elucidate the subjects treated of—believing that by this means the interests of his readers would be much more effectually consulted than by offering them imperfect information with a trifling diminution in the size of the volume; especially as the subscribers to the first volume were not considered as engaging for both.

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CHAPTER I.

Of the proportion of variable quantities.

ARTICLE 1. Definition. If two variable quantities be so connected that the ratio between any two values of one of them is always the same as the ratio between the corresponding values of the other, or in other words, that one cannot increase or decrease, but the other must increase or decrease in the same ratio; either of the quantities is said to vary directly as the other, or, more shortly, to be as the other.

Thus, if A and B be mutually dependent upon each other in such a way that if A be changed to any other value, a , B must be changed to another value, b such that $A : a :: B : b$, A is said to vary directly as B , or B to vary as A .

EXAMPLE.

If money be divided equally among a certain number (n) of persons, each persons share will vary as the sum divided.

A

For

For each person receives an n th part of the whole sum $\therefore$ each persons $share = \frac{sum}{n}$, and

Share in one case : share in any other case ::

$\frac{\text{Sum in the one case}}{n} : \frac{\text{Sum in the other case}}{n}$

$\therefore$ *Sum in the one case : Sum in the other case,*
Or share varies as the sum.

2. Definition. If two variable quantities be so connected, that the ratio between any two values of one of them, is always the same as the inverse ratio of the corresponding values of the other, or in other words, that one cannot increase or decrease, but the other must decrease or increase in the same ratio; the one quantity is said to vary inversely as the other.

Thus A varies inversely as B , if when A is changed to a , B must be changed to b , such that $A : a :: b : B$.

Cor. If one quantity varies inversely as a second, that second will vary inversely as the first. for if A varies inversely as B , or $A : a :: b : B$; $B : b :: a : A$, that is, the ratio between the two values, B and b , is equal to the inverse ratio between the two corresponding values, A and a $\therefore$ (by definition) B varies inversely as A .

EXAMPLE.

If a certain sum of money (S) be equally divided

vided among people, each person's share varies inversely as the number of persons.

$$\text{For a share} = \frac{S}{\text{Number of persons}} \therefore$$

a share in one case : a share in any other case ::

$$\frac{S}{N^{\circ} \text{ of persons in the one case}} : \frac{S}{N^{\circ} \text{ in the other case}} :: N^{\circ} \text{ of per. in the other case} : N^{\circ} \text{ in the former case,}$$

Or share varies inversely as the number of persons.

3. The sign $\propto$ placed between two quantities denotes that one varies as the other. for example, if x varies as y , it is expressed thus $x \propto y$. In the following articles, z, y, x , &c. are used to denote variable quantities. z, y, x , &c. to denote any fixed values of x, y, z , &c. corresponding with each other; and z, y, x , &c. any other such fixed values; that is, when the value of z , is z , the values of y, x , &c. are supposed to be y, x , &c. and when the value of z , is z , the values of y, x , &c. are supposed to be y, x , &c.

4. If one quantity varies *inversely* as another, it varies *directly* as the reciprocal of that other. For if x varies inversely as y , $x : x :: y : y$, $x : x :: \frac{1}{y} : \frac{1}{y}$ i. e. the ratio between any two values of x (x and x) is equal to the ratio between the reciprocals ($\frac{1}{y}$ and $\frac{1}{y}$) of the corresponding values (y and y) of y ; $\therefore$ by Art. 1st, $x \propto \frac{1}{y}$.

5. Conversely, if one quantity varies *directly* as the reciprocal of another, it varies *inversely* as that other. For if $x \propto \frac{1}{y}$ or $x : x :: \frac{1}{y} : \frac{1}{y}$ $x : x ::$

$y : y$ $\therefore$ by Art. 2, x varies inversely as y .

6. If one quantity varies as a second, and that second as a third, the first varies as the third. For let $x \propto y$, and $y \propto z$, then since $x : x :: y : y$ and $y : y :: z : z$, $x : x :: z : z$, or $x \propto z$.

7. If one quantity varies as a 2d. and that 2d. varies inversely as a 3d. the 1st. varies inversely as the 3d. For if $x \propto y$ and y varies inversely as z , (Art. 4) $y \propto \frac{1}{z} \therefore$ (Art. 6) $x \propto \frac{1}{z} \therefore$ (Art. 5) x varies inversely as z .

8. If one quantity varies inversely as a 2d. and that 2d. varies as a 3d. the 1st. varies inversely as the 3d. For let x vary inversely as y , and $y \propto z$; then $x : x :: y : y$ and $y : y :: z : z$ $\therefore$ invert. $y : y :: z : z \therefore x : x :: z : z$, or x varies inversely as z .

9. If one quantity varies inversely as a 2d. and that 2d. varies inversely as a 3d. the 1st. varies directly as the 3d. For let x vary inversely as y and y vary inversely as z ; then

$x : x :: y : y$ and $y : y :: z : z \therefore y : y :: z : z$ $\therefore x : x :: z : z$ or $x \propto z$.

10. If one quantity varies as another, it will vary as that other multiplied or divided by any constant quantity whatsoever. For any proposed multiple

multiple part or parts of a quantity, changes in the same ratio as the quantity itself. Thus, if $x \propto y$, $x \propto my$ (m being constant) for

$$x : y :: my : my \therefore x : x :: my : my, \text{ or } x \propto my.$$

11. When three variable quantities (X, Y, Z ,) are so connected that no change takes place in the 1st. unless some change takes place in one or both of the others, and no change can take place in either of those others, but the value of the 1st. changes in the same ratio, and if the others be both changed, the 1st. is proportionably changed on both accounts (so that, for example, the three having at first certain values, if Y be doubled, X is doubled; if Y be doubled and Z be tripled, X becomes triple what it would have been, had Y alone been changed; so as on the whole to be increased six fold) the first varies as the product of the two others, and is said to vary as those others jointly.

Thus, if a subscription be made, each person subscribing an equal number of pounds, the sum subscribed, which, agreeably to the above hypothesis, changes in the same ratio as the number of subscribers, and in the same ratio as the magnitude of a share, varies as the number of subscribers, $\times$ number of pounds constituting a share.

Let S denote the sum when N persons subscribe P pounds; and s the sum when n persons subscribe p pounds; then it is to be proved that $S : s :: N \times P : n \times p$.

Now

Now sum (S) when N persons subscribe P pounds: sum when n persons subscribe P pounds :: $N : n$

and sum when n persons subscribe P pounds : sum (s) when n persons subscribe p pounds :: $P : p$;
 $\therefore$ ex æquo $S : s :: NP : np$.

In general let x be the value of X , when the values of Y and Z are y and z ; x when they are, y and z ; x'' when y' and z' ; then it is to be demonstrated that

$x : x :: y z$ the product of those values of Y and Z which correspond with x ; $y' z'$, the product of those values of Y and Z , which correspond with x'' .
 Now by hypothesis.

$$x : x :: y : y'$$

$$\text{and } x : x :: z : z'$$

$$\therefore \text{ex æquo } x : x :: y z : y' z' *$$

The hypothesis of the above proposition may be briefly stated as follows:

When Y is $\dagger$ given $X \propto Z$

When Z is given $X \propto Y$.

Hence in any case, where this statement can be applied, it is inferred that when $\dagger$ neither are given $X \propto Y Z$.

* Aliter. Let y be changed to ny ; and by hypothesis, x changes to $n x$; then let z be changed to $m z$, and by hypothesis $n x$ changes to $m n x$; but $x : m n x :: y \times z : n y \times m z$, that is, x is changed in the same ratio as $y z$.

† That is, remains the same in different cases.

‡ It would be better perhaps to say that generally, whether one or both be given, $X \propto Y Z$.

EXAMPLE.

Thus, because when the number of subscribers is given, $\text{sum} \propto \text{share}$, and when the amount of a share is given, $\text{sum} \propto N^{\circ} \text{ of subscribers}$, it may be inferred that in general

$$\text{Sum} \propto \text{share} \times N^{\circ} \text{ of subscribers.}$$

12. If one variable quantity be so connected with three others, that when any two of those three remain constant, it varies as the remaining third, *in general* it varies as the product of the three, and is said to vary as the three jointly. For let X be the first quantity; and W, Y, Z , the three on which it so depends; and let the following be any four sets of corresponding values

x	w	y	z
$\overset{1}{x}$	$\overset{1}{w}$	$\overset{1}{y}$	$\overset{1}{z}$
$\overset{2}{x}$	$\overset{2}{w}$	$\overset{2}{y}$	$\overset{2}{z}$
$\overset{3}{x}$	$\overset{3}{w}$	$\overset{3}{y}$	$\overset{3}{z}$
$\overset{4}{x}$	$\overset{4}{w}$	$\overset{4}{y}$	$\overset{4}{z}$

then by hypothesis,

$$x : \overset{1}{x} :: w : \overset{1}{w}$$

$$\text{and } x : \overset{2}{x} :: y : \overset{2}{y}$$

$$\text{and } x : \overset{3}{x} :: z : \overset{3}{z}$$

$$\therefore \text{ex æquo } x : x :: w y z : w y z$$

that is, the ratio between any two values of X , is the same as the ratio between the corresponding values of $W Y Z$, or $X \propto W Y Z$.

EXAMPLE.

The (simple) interest due in different cases for money lent varies as the sum lent $\times$ the rate per cent. $\times$ the time it has lain at interest. For if the rate and the time be given, it varies as the sum; if the rate and sum be given, it varies as the time; and if the sum and time be given, it varies as the rate.

13. In the same manner it may be shewn that if one quantity be so connected with four or more others, as to vary as each of those others when the rest of them are constant, * it will vary as their product; and is said to vary as the same jointly.

14. If three variable quantities be so connected, that the first is given when both the second and third are given, but varies, when they vary; directly as the second, and inversely as the third, (so that for example, if the second be tripled, the value of the first is tripled, and if the second be tripled and the third doubled, the first is tripled on the one account and halved on the other, so as

* If it be considered that the magnitude of a product depends on the magnitude of the factors—that no factor can be changed but the product is changed in the same ratio—so that for example, if one factor be halved, the product is halved—if one factor be halved and another quartered, the product is reduced to half a quarter of its former value—if three factors be each tripled, the product is increased 27 fold—and the like in other instances, it will clearly appear why a magnitude, whose changes are directly proportional to the changes in certain other magnitudes, must always vary as the product of those others.

upon

VARIATION.

9

upon the whole, to become three halves of it's former value,) the first varies as the $\frac{\text{second}}{\text{third}}$

For let x vary directly as y , and inversely as z ; then x varies directly as y , and directly as $\frac{1}{z}$ according to the hypothesis of article 11, $\therefore$ (by that art.) $x \propto y \times \frac{1}{z}$ or $\frac{y}{z}$.

EXAMPLE.

The amount of a share $\propto \frac{\text{sum subscribed}}{\text{N}^{\circ} \text{ of subscribers.}}$

For if the number of subscribers be given, a share $\propto$ sum subscribed; and if the sum subscribed be given, a share varies inversely as the number of persons.

15. And in the same manner it may be shewn, that if one quantity so depends on any number whatsoever of others, that when all but one of them are given, it varies either directly or inversely as the remaining one, it varies as a fraction whose numerator is the product of those whose variation is direct, and denominator the product of those whose variation is inverse. For example, if x so depends on p, q, r, s , and t , that when q, r, s , and t are given $x \propto p$; when p, r, s , and t are given, $x \propto q$; when p, q, s and t are given, $x \propto r$; when p, q, r , and t are given, x varies inversely as s , and when p, q, r , and s are

B
given,

given, x varies inversely as t ; x will vary

as $\frac{p \times q \times r}{s \times t}$.

16. If one quantity varies as another, it is equal to that other multiplied into some constant quantity. For let $x \propto y$, and let a and b be two co-temporary values of x and y ; then $x : y :: a : b \therefore x = y \times \frac{a}{b}$.

17. If one quantity varies as another, their sum or difference will vary as either of them. For

let $x \propto y$; then $\overset{1}{x} : \overset{1}{y} :: \overset{1}{x} : \overset{1}{y}$

and $\overset{1}{x} \pm \overset{1}{y} : \overset{1}{y} :: \overset{1}{x} \pm \overset{1}{y} : \overset{1}{y}$

$\therefore \overset{1}{x} \pm \overset{1}{y} : \overset{1}{x} \pm \overset{1}{y} :: \overset{1}{y} : \overset{1}{y}$

or the ratio between any two values of $x \pm y$ is the same as the ratio between the corresponding values of y ; i. e. $x \pm y \propto y$.

18. If three quantities vary as each other, any combination of them in which they are connected by the signs $+$ or $-$ will vary as any one of them. For let x, y and z vary as each other; then by the last art. $\overline{x \pm y} \propto z \therefore$ by same art. $\overline{x \pm y \pm z} \propto z \propto y \propto x$.

19. In the same manner if there be more than three quantities that vary as each other, it may be shewn that all or any part of them connected any way by the signs $+$ or $-$ will vary as any one of them.

20. And

20. And hence any one of such combinations will vary as any other of them.

21. If one quantity varies as another, their like powers or roots will vary as each other. For let $x \propto y$; and let n denote any number whole or fractional, positive or negative; then since

$$x : x^n :: y : y^n, \quad x^n : x^n :: y^n : y^n$$

$$\text{or } x^n \propto y^n,$$

22. If one quantity varies as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products or quotients will vary as each other. Let $x \propto y$, then $xz \propto yz$. For since

$$x : x :: y : y$$

$$\text{and } z : z :: z : z$$

$$xz : xz :: yz : yz$$

$$\text{and } \frac{x}{z} : \frac{x}{z} :: \frac{y}{z} : \frac{y}{z}$$

That the proposition is true when the multiplier or divisor is constant, appears also from art. 10.

Aliter. Let $x \propto y$ and $= my$; then $xz = myz \propto yz$ (art. 10.)

23. Cor. 1. If $z = y$ the last proportion be-

comes $\frac{x}{y} : \frac{x}{y} :: \frac{y}{y} : \frac{y}{y}$; and (since $\frac{y}{y} = \frac{y}{y}$)
 $\frac{x}{y} = \frac{x}{y}$; or all the values of $\frac{x}{y}$

are the same, i. e. If one quantity varies as another, their quotient is constant.

This cor. is usually inferred as follows;

Let $x \propto y$; then $\frac{x}{y} \propto \frac{y}{y} \propto 1$, or is constant.

Aliter. Let $x \propto y$ and $= my$; then $\frac{x}{y}$ always $= m$.

24. Cor. 2. *Conversely*, if a fraction having a variable numerator and denominator be invariable, the numerator varies as the denominator.

25. Cor. 3. If one quantity varies inversely as another, their product is constant.

For if x varies inversely as y , $x \propto \frac{1}{y} \therefore$ by

Cor. 1. ($x \div \frac{1}{y}$, or) $x \times y$ is constant.

26. Cor. 4. *Conversely*, if the product of two variable quantities be invariable, those quantities vary inversely as each other.

27. Cor. 5. If one quantity varies as two others jointly, either of the latter will vary as the former directly, and the other of the two inversely.

inversely. For let $x \propto y z$, then, dividing both sides by y , $z \propto \frac{x}{y}$, dividing both sides by z , $y \propto \frac{x}{z}$

28. *In general*; if two quantities vary as each other, any of the factors may be transferred by division from the numerator of one of them, to the denominator of the other, and vice versa by multiplication: and the results, if they be variable, will still vary as each other,—but if one side be constant, the other side will be constant.

Ex. Let $t v \propto x y z$; then $\frac{t v}{x} \propto y z$;

$t \propto \frac{x y z}{v}$; $z \propto \frac{t v}{x y}$; $\frac{x y z}{t v} \propto 1$, or is constant; &c. &c.

29. If one quantity varies as a 2d, and a 3d as a 4th, the product or quotient of the 1st, and 3d, will vary as the product or quotient of the 2d and 4th.

Let $x \propto y$, and $w \propto z$; then $x w \propto y z$, and $\frac{x}{w} \propto \frac{y}{z}$.

For since $\overset{1}{x} : \overset{2}{x} :: \overset{1}{y} : \overset{2}{y}$

and $\overset{1}{w} : \overset{2}{w} :: \overset{1}{z} : \overset{2}{z}$

$\overset{1}{x} \overset{1}{w} : \overset{2}{x} \overset{2}{w} :: \overset{1}{y} \overset{1}{z} : \overset{2}{y} \overset{2}{z}$

and $\overset{1}{x} : \overset{2}{x} :: \overset{1}{y} : \overset{2}{y}$
 $\overset{1}{w} : \overset{2}{w} :: \overset{1}{z} : \overset{2}{z}$

30. In

30. In a similar manner may any number of proportions of this kind be combined by multiplication or division.

Ex. Let $y \propto z$, and $w \propto x$, and $v \propto u$;

then $y w v \propto z x u$; $\frac{yw}{v} \propto \frac{zx}{u}$; &c. &c.

31. Cor. If two quantities vary each as a third, the square root of their product will vary as that third.

Let $x \propto z$, and $y \propto z$; then $xy \propto z^2$

$\therefore$ (art. 21) $\sqrt{xy} \propto z$.

If three quantities vary each as a fourth, the cube root of their product will vary as that fourth.

Let $x \propto z$, and $y \propto z$, and $w \propto z$;

then $xyw \propto z^3$ $\therefore$ (art. 21) $\sqrt[3]{xyw} \propto z$.

In general; if n quantities vary each as the same, the n^{th} root of their product will vary as that same.

SCHOLIUM.

Variation is the equality of ratios, To say that $x \propto y$ is saying that the ratio between any two values of x is equal to the ratio between the corresponding values of y ; (see definitions) or, agreeably to the notation used above, that

$$\left(\begin{smallmatrix} 1 \\ x : x \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 \\ y : y \end{smallmatrix} \right). \quad \text{If } x \propto \frac{y}{z}$$

it is that $(x : x) = \left(\frac{y}{z} : \frac{y}{z} \right)$ or

$$\left(\frac{y}{z} : \frac{y}{z} \right) + (z : z) \text{ or } (y : y) - (z : z).$$

$$\text{If } x \propto \frac{1}{y} (x : x) = (1 : 1) - (y : y) =$$

$$(\text{since } 1 : 1 \text{ is } 0) - y : y. \quad \text{If } x y \propto \frac{z}{w}$$

$$(x : x) + (y : y) = (z : z) + (z : z) \\ + (z : z) - (w : w) = 3 \times (z : z) - (w : w)$$

&c. &c. see ratios. And by keeping more closely to this method of considering the subject, many of the above propositions may be exhibited in a view apparently different, and somewhat shortened: e. g. article 22, only asserts that if two ratios are equal, and equal ratios be added to or subtracted from each, the sums or remainders will

be equal. If $x \propto y$, i. e. if $(x : x) = (y : y)$

adding the ratio $(z : z)$ to both sides,

$$(x : x) + (z : z) = (y : y) + (z : z), \text{ i. e.}$$

$$(xz : xz) = (yz : yz) \text{ or } xz \propto yz.$$

If $S \propto TV$, i. e. if $(S : S) = (T : T)$

+ $(V : V)$ subtracting the ratio $(T : T)$

from each side $(S : S) - (T : T) = (V : V)$

$$\text{i. e. } \left(\frac{S}{T} : \frac{S}{T} \right) = (V : V) \text{ or } \frac{S}{T} \propto V. \text{ \&c. \&c.}$$

But

But this method was avoided from considering the propensity observable in many minds, of not resting contented with the matter of fact and operations, denoted by these technical expressions, but searching for something abstract, as they call it, such as the notion of ratio being quantity *fui generis*; and the like; notions, which though merely the creatures of definition and compact, are yet by a strange confusion of words and ideas, considered by them as indicating some real independent truth.

APPENDIX,

OF THE RULE OF THREE AND REDUCTION.

32. On the doctrine of proportion, and Variation is founded the *Rule of Three* in all its branches. If the name of *cause* and *effect* be given to those things, whose magnitudes depend directly on each other, e. g. if the quantity of goods bought be called the cause of the price; the time and number of men, the joint causes of work performed; &c. then in all questions belonging to the simple rule of three direct, there will be found a *cause* and an *effect* given, and it is either required what *effect* another stated *cause* will produce, or it is required what *cause* is necessary to the production of another

ther

ther stated *effect*.—And in all questions belonging to the rule of three inverse, or to compound proportion, there will be found an *effect* given as being produced by the joint agency of certain given *causes*, and it is either required what *effect* the joint agency of certain other given *causes* of the same kind will produce, or, another *effect* being given, and all the *causes* producing it but *one*, it is required what that *one* must be in order that, jointly with the *rest*, it may produce that *effect*. Now, because the *effect* (by hypothesis) varies as the *cause*, or *causes* jointly, i. e. as the *product* of all the *agents* forming the *cause*, therefore any question is to be resolved by putting the *Product* of the *agents* forming one *cause*, to the *product* of the *agents* forming the other *cause*, as the *effect* produced by the former *cause*, to the *effect* produced by the other.

Note, The cause and effect which are wholly given, are called the *supposition*; and the cause and effect in which the *number sought* is found, is called the *demand*.

EXAMPLE TO SIMPLE RULE OF THREE DIRECT.

If a man can count over 150 guineas in 2 minutes, how many minutes would he be, at the same rate, in counting over 1,000,000 guineas.

The time is caused by the number of guineas—twice as many requiring twice the time—half as many, half the time; &c. i. e. the *time* varies as the *number* of guineas.

$\therefore 150 \text{ guineas} : 1000,000 \text{ guineas}$
 $:: 2 \text{ minutes} : \text{Number of minutes required, which}$
 is therefore equal to

$$\frac{1000,000 \text{ guineas}}{150 \text{ guineas}} \times 2 \text{ minutes}$$

$$= \frac{1000000}{150} \times 2 \text{ minutes}$$

$$= 13333 \text{ minutes } \& \frac{1}{3} \text{ of a minute.}^*$$

EXAMPLES OF CASES WHERE THE EFFECT IS
 PRODUCED BY JOINT AGENCY.

Ex. 1. If 100 workmen can finish a piece of work in 12 days, how many are sufficient to do the same in 3 days.

The performance of the work is caused by the men and time jointly, i. e. *quantity of work performed* $\propto$ *number of men* $\times$ *number of days*, for if the number of days be given, it varies as the number of men; and if the number of men be given, it varies as the number of days.

$\therefore$ the quantity of work in one case, is to the quantity of work in the other, as 100×12 , to $3 \times$ the number sought; but the quantity of work is supposed to be the same in both cases, or the first term of the proportion to be equal to the second;

$$\therefore 100 \times 12 = 3 \times \text{number sought and number sought} = \frac{100 \times 12}{3} = 400.$$

* Equal to 9 days, 6 hours, 13 minutes, 20 seconds.

Ex.

RULE OF THREE.

19

Ex. 2. If 100 workmen can finish a piece of work in 12 days, how many are sufficient to finish five times as much in 3 days.

Here the *quantity of work in one case*, is to the *quantity in the other*, as 1 to 5 $\therefore 1 : 5 :: 100 \times 12 : 3 \times \text{Number sought} \therefore \text{Number sought} = \frac{100 \times 12 \times 5}{3 \times 1} = 2000.$

The former of these examples is said to belong to the *Rule of three inverse*, because if the equation $100 \times 12 = 3 \times \text{number sought}$ be thrown into a proportion, it will be, 100 days : 3 days (not as (12) the number of men employed the 100

C 2
days,

Every case of compound and inverse proportion may be solved by two or more simple direct proportions. For instance, in the last example we may inquire, 1st. what number of men can perform 5 pieces of work, if 100 men can perform one; (supposing the time 12 days in both cases) stated thus, $1 : 5 :: 100 : \text{number sought} = 500.$

2. If 5 pieces of work are performed in 12 days, how many in 3 days; (supposing the number of men 500 in both cases) stated thus, $12 : 3 :: 5 : \text{number sought} = \frac{15}{12}.$

3. If $\frac{15}{12}$ of the work requires 500 men, how many men will $\frac{5}{12}$ pieces require; (supposing the time in both cases three days) stated thus, $\frac{15}{12} : 5 :: 500 : \text{The final number sought} = 500 \times 5 \div \frac{15}{12} = 2000.$

And every question being thus reduced to direct simple statements, the truth of the operations in all cases may be made evident, without referring back to any principle of ratios. For allowing that double
the

days, to the number of men employed the 3 days, but *inversely*) :: *Number sought* : 12; and this inversion, it is evident, will always happen when the two effects are the same.

The 2d Example, and others similar, are said to belong to compound proportion—In a scientific arrangement, inverse proportion must be considered as a particular case of compound.

the quantity of goods costs twice as much, half, half as much, $\frac{m}{n}$ ths, $\frac{m}{n}$ ths as much; with similar concessions in other cases, let the

3 terms of a simple direct statement be A, a, B , — A & B being the terms of the supposition; then, because the number A of a certain denomination costs (if it be concerning worth of goods) $B, 1$, which is only an A th part of A , will cost an A th part of B ; but if one costs an A th part of B, a will cost a A th parts of B — or at once we may infer, that if A costs B, a , being an $\frac{a}{A}$ th part of A , must cost

$\frac{a}{A}$ th of B . — i. e. the term sought will be $\frac{a}{A} \times B$. and if instead of “costs” we substitute the word “requires” the reasoning becomes general.

For example, if six yards of muslin cost thirty three shillings, how much will eleven yards cost?

1. If 6 yards cost thirty-three shillings, 1 yard will cost a 6th part of thirty-three shillings; or $\frac{33}{6}$ shillings (= 5s. 6d.)

2. If one yard costs $\frac{33}{6}$ s. eleven yards will cost eleven times $\frac{33}{6}$ s. or $\frac{11 \times 33}{6}$ s. (= 3l. 0d. 6d.) the same answer as would arise from the statement, 6yds. : 11yds :: 33s. *Number of sh. sought.*

33. It is obvious that in all cases whatsoever the *term sought* is equal to the *given term of the same kind*, (which for distinctions sake may be called the homologous term) *multiplied into a fractional expression, whose numerator is the product of some of the other given terms, and denominator the product of the rest*, and because the increase of any factor in the numerator of a fraction increases the value of the fraction, but the increase of any factor in the denominator decreases its value, it follows that any term in the question, which, had it been given greater, *cæteris paribus*, would have caused the term sought to have come out greater, must be a factor of the numerator—and that any term, which had it been given greater, *cæteris paribus*, would have caused the term sought to have come out less, must be a factor of the denominator;—which forms an easy rule for solving all questions whatsoever without any statement—it being only necessary to inquire which of the given terms the number sought varies *directly* as, and which it varies *inversely* as.—Let us take the last question for an example.

1. If instead of * *one* piece of work, the 100 men had finished *more* in 12 days, it would have required *fewer* men to finish five pieces in three days; therefore the 1 is a factor of the denominator.

* It is only in order to shew the method, that the place of this term is considered—for it makes no difference whether *unity* be a factor of the numerator or of the denominator.

2. If

2. If the 100 men, instead of being 12 days, had been *more* days, it would have taken *more* men to finish the 5 pieces in 3 days; $\therefore$ 12 is a factor of the numerator.

3. If it had been required how many men could finish a *greater* quantity than 5 pieces, the answer would have come out *greater*; $\therefore$ 5 is a factor of the numerator.

4. If instead of 3 days, *more* days had been allowed, *fewer* men would have been required; $\therefore$ 3 is a factor of the denominator, and the *number sought* is $100 \times \frac{12 \times 5}{1 \times 3}$ as before.

34. But this rule may be shortened from this consideration; that no two of the given quantities, that are of the same kind, can both belong to the numerator, or both belong to the denominator; as easily appears if the question be resolved by simple statements, since in a simple statement,

$$A : A :: N : \text{number sought, (where } A \text{ and } A \text{ are supposed to be of the same kind, and } N \text{ of the same kind as that, whose number is sought) number}$$

$$\text{sought} = \frac{N \times A}{A}$$

It is sufficient therefore, omitting the supposition, to inquire concerning the given terms in the demand, which of them vary directly, and which inversely as the number sought—placing the former in the numerator, and the latter in the denominator;

nator; and placing those terms of the supposition in the denominator, whose homologous terms in the demand fell to the numerator; and vice versa.

EXAMPLE.

If b horses eat up c acres of grass in d days, in how many days will e horses eat up f acres?

The number of days varies directly as the quantity of pasture, and inversely as the number of horses—therefore f is a factor of the numerator; and e a factor of the denominator; hence according to the above observation, c belongs to the denominator, and b to the numerator $\therefore$ number

$$\text{sought} = d \times \frac{fb}{ec}.$$

According to the other method of cause and effect, the solution would be as follows.

The horses and time are the joint causes of the quantity eaten, or *quantity eaten* $\propto$ *number of horses* $\times$ *number of days* $\therefore$

$$c \text{ acres} : f \text{ acres} :: b \times d : e \times \text{number sought}$$

$$\therefore \text{number sought} = \frac{fbd}{ec} \text{ or } d \times \frac{fb}{ec} \text{ as before.}$$

35. The operation, by which we find the quantity sought in any question, is carried on by multiplication and division of numbers, which are in the ratio of the given quantities; therefore if any term contains quantities of different denominations, such term must first be reduced to an equivalent

equivalent expression containing but one denomination, and each term must be made of the same denomination as its homologous term.

Ex. What is the tax upon 13ol. 14s. at 5s. 6d. in the pound?

Here 1l. : 13ol. 14s. :: 5s. 6d. : *money sought* ;
which $\therefore$ is equal to $\frac{13ol. 14s.}{1l.}$ of 5s. 6d. meaning

by $\frac{13ol. 14s.}{1l.}$ the number expressive of how many

times 1l. is contained in 13ol. 14s. and in order to find the numerical value of this fraction, we must reduce 13ol. 14s. and 1l. to shillings—they become

2614sh. and 20sh. $\therefore$ *money sought* is $\frac{2614}{20}$ ths of

5s. 6d. or (since 5s. 6d. = 66d.) $\frac{2614}{20}$ ths of 66d

$$= \frac{2614 \times 66}{20} \text{ pence} = \frac{2614 \times 33}{10} \text{ pence} = \frac{86262}{10}$$

$$\text{pence} = 8626 \text{ pence} + \frac{\text{penny}}{5}.$$

In order to facilitate the *practice* of thus reducing quantities from one denomination to another, the following rules are subjoined,

36. *Definition.* The number denoting how many of one denomination are contained in an integer of another, is called the *Aliquotient* of the two denominations.

Thus, 12 is the *aliquotient* of *pence* and *shillings*—one *shilling* containing 12 *pence*.

37. To reduce a quantity from a higher to a lower denomination.

Rule. Multiply the given quantity by the aliquotient of the two denominations.

Ex. 1. In 25*l.* how many *crowns*. Answer, 4×25 or 100.

Ex. 2. In 17 *yards* how many *inches*?

A yard contains 36 inches. Answer 36×17 or 612.

38. *Converse.* To reduce a quantity from a lower to a higher denomination.

Rule. Divide by the aliquotient of the two denominations—The integral quotient is the number of units of the higher denomination contained in the given quantity. The remainder, if there be any, shews how many of the lower are contained in it besides.

Ex. 1. In 100 *crowns* how many *pounds*. Answer $100 \div 4$ or 25.

Ex. 2. In 57 *pence* how many *shillings*.

The quotient of 57 divided by 12 is 4; the remainder 9; $\therefore$ the answer is 4*sh.* 9 *pence* over.

$$\begin{array}{r} 12 \overline{) 57} 4 \\ \underline{48} \\ 9 \text{ rem.} \end{array}$$

39. To reduce a quantity containing one or more denominations, to an equivalent expression in the lowest denomination found in it, or to any inferior to that.

Rule. Reduce that part of the given quantity, which is of the highest denomination, to the next inferior or 2d denomination; adding that part which is of the 2d denomination. Reduce the result to the 3d denomination, adding what is of the 3d. Proceed thus through all the denominations, and the number last found will be the expression required.

Ex. 1. Reduce 3*l.* 11*sh.* 2 *far.* to farthings.

$$\begin{array}{r} \text{3l. + 11sh.} = 3 \times 20 + 11 \\ (71) \text{ shillings} \end{array} \quad \begin{array}{r} \text{£. s. far.} \\ 3 \quad 11 \quad 2 \\ \underline{20} \\ 71 \\ \underline{12} \\ 852 \\ \underline{4} \\ 3410 \text{ farthings} \end{array}$$

$$= 71 \times 12 (852) \text{ pence}$$

$$\therefore 3\text{l.} + 11\text{s.} + 2\text{f.} = 852 \text{ pence} + 2 \text{ farthings}$$

$$= 852 \times 4 + 2 (3410) \text{ farthings.}$$

Ex.

REDUCTION.

27

Ex. 2. How many *inches* are contained in a *mile*?

N. B. A mile contains 1760 yards. 1760

$$\begin{array}{r} 3 \\ \hline 5280 \text{ feet} \\ 12 \end{array}$$

Answer 63360 inches.

Ex. 3. In 13l. 10s. how many *groats*?

$$\begin{array}{r} \text{£. s.} \\ 13 \quad 10 \\ 20 \\ \hline 270 \text{ shillings} \\ 3 \\ \hline \end{array}$$

Answer 810 *groats*.

40. *Converse.* To reduce a quantity of one denomination, to its value in the higher denominations.

Rule. Reduce the given quantity by division, to the next higher denomination, (art. 38) and set by the remainder. Reduce the quotient to the next higher still, setting by the remainder as before—Proceed thus, so far as integral quotients can be obtained—The last quotient, together with the several remainders, form the answer; the remainders being of the same denomination as the dividends they arise from.

D 2

Ex.

Ex. Reduce 3410 farthings to pounds, shillings, &c.

$$\begin{array}{r}
 4 \overline{) 3410} \text{ farthings} \\
 12 \overline{) 82} \text{ . } \text{---} 2 \text{ rem}^r \\
 20 \overline{) 71} \text{ . } \text{---} 0 \\
 3 \text{ . } \text{---} 11
 \end{array}$$

Answer 3*l.* 11*s.* 0*d.* 2*far.*

41. To find the value of a fraction of any given denomination in terms of the lower denominations.

Rule. Consider the numerator as a quantity of the given denomination—Reduce it to a lower denomination, till it be not less than the denominator. Divide the result by the denominator—Set by the quotient. Reduce the remainder, if there be any, to a denomination still lower—Divide and proceed as before, till there be either no remainder, or no lower denomination.—The several quotients form the answer; and the remainder after the last division, (if there be any) divided by the divisor, shews what fractional part of an integer of the lowest denomination the given quantity contains besides.

Ex.

REDUCTION.

29

Ex. I. Required the value of $\frac{3}{7}$ ths of 1l.

$$\begin{array}{r} 3 \\ 20 \\ 7 \overline{)60} \\ \underline{8} \end{array} \dots 4 \text{ rem}^r.$$

$$\begin{array}{r} 12 \\ 7 \overline{)48} \\ \underline{6} \end{array} \dots 6 \text{ rem}^r.$$

$$\begin{array}{r} 4 \\ 7 \overline{)24} \\ \underline{3} \end{array} \dots 3 \text{ rem}^r.$$

Answer 8s. 6d. 3far. + $\frac{3}{7}$ of a farthing.

$$\frac{3}{7} \text{ of a pound} = \frac{3 \times 20}{7} \text{ of a shilling.}$$

$$= \frac{60}{7} \text{s.} = 8\text{s.} + \frac{4}{7}\text{s.}$$

$$\frac{4}{7}\text{s.} = \frac{4 \times 12}{7} \left(= \frac{48}{7} = 6 + \frac{6}{7} \right) \text{pence;}$$

$$\frac{6}{7} \text{pence} = \frac{4 \times 6}{7} \left(\frac{24}{7} = 3 + \frac{3}{7} \right) \text{farthings}$$

$$\therefore \frac{3}{7}\text{l.} = 8 + \frac{4}{7}\text{sh.} = 8\text{sh.} + 6 + \frac{6}{7} \text{pence}$$

$$= 8\text{s.} + 6\text{d.} + 3 + \frac{3}{7} \text{farthings.}$$

Ex.

REDUCTION.

Ex. 2. Required the value of $\frac{13}{64}l$.

$$\begin{array}{r}
 13 \\
 20 \\
 \hline
 64 \overline{)260} (4 \text{ quotient.} \\
 \underline{256} \\
 4 \text{ rem}^r. \\
 12 \\
 \hline
 48 \\
 4 \\
 \hline
 64 \overline{)192} (3 \text{ quotient.} \\
 \underline{192} \\
 \hline
 \text{..}
 \end{array}$$

Answer 4s. od. 3far.

Ex. 3. Required the value of $\frac{5}{12}$ of 7 guineas.

$$\frac{5}{12} \text{ of 7 guineas} = \frac{5 \times 7}{12} \text{ of a guinea.}^*$$

$$\begin{array}{r}
 35 \\
 21 \\
 \hline
 35 \\
 70 \\
 \hline
 12 \overline{)735} \\
 \underline{61} \quad - \quad - \quad 3 \text{ rem}^r. \\
 12 \\
 \hline
 12 \overline{)36} (\\
 \underline{3}
 \end{array}$$

Answer 6s. 3d.

* If $\frac{5}{12}$ be considered as the fraction, seven guineas must be the denomination, not one guinea, and consequently the first multiplier must be 7×21 instead of 21.

Ex.

REDUCTION.

31

Ex. 4. Required the value of $\frac{2}{3}$ of 4 miles

$$\frac{2}{3} \text{ of } 4 \text{ miles} = \frac{8}{3} \text{ of a mile.}$$

$$\begin{array}{r} 3 \overline{)8} \\ \underline{6} \\ 2 \text{ rem}^r \\ 1760 \\ 3 \overline{)3520} \\ \underline{3000} \\ 520 \\ \underline{480} \\ 40 \text{ rem}^r \end{array}$$

Answer 2 miles, 1173 yards, 1 foot.

42. To find the value of any given quantity in terms of any proposed quantity.

Rule. Reduce the two quantities to equivalent expressions of a common denomination. The expression for the given quantity divided by that for the proposed quantity is the value required.*

Ex. 1. What fraction of a shilling is 3 pence.

Answer $\frac{3}{12}$. For one penny is one 12th $\therefore$ 3 pence is 3 12ths.

Ex. 2. What fraction of a guinea is half a crown?

Half a crown contains 5 six-pences; and a guinea contains 42 $\therefore$ Half a crown = $\frac{5}{42}$ of a guinea.

* For, to find B in terms of C , we must find the value of the fraction $\frac{B}{C}$, since $B = \frac{B}{C} \times C$, or B is that part or parts of C , expressed by the fraction $\frac{B}{C}$; and this value is found by the rule.

Ex.

Ex. 3. What part of 2d. is $\frac{2}{3}$ of 7d?

Answer $\frac{2}{3} \times 7 \div 2 = \frac{7}{3}$.

Ex. 4. What part of a pound is 6s. $8\frac{1}{2}$ d?

6s. $8\frac{1}{2}$ d. contains 161 halfpence; and 1l. contains

480 halfpence $\therefore$ the answer is $\frac{161}{480}$

Ex. 5. What fraction of a foot is $\frac{5}{7}$ of an inch?

Answer $\frac{5}{7} \div 12$ or $\frac{5}{84}$

Ex. 6. What fraction of one inch is $\frac{5}{84}$ of a foot?

Answer $\frac{5 \times 12}{84} \div 1$ or $\frac{5}{7}$.

Ex. 7. What parts of 1l. is 3l. 7s.

3l. 7s. = 67s. Answer $\frac{67}{20}$.

DECIMAL REDUCTION.

43. To find the value of a decimal of any denomination, in integral terms of the lower denominations.

Rule. Reduce the decimal to its value in the next lower denomination, by multiplying it into the aliquot of that and the given denomination. In the same manner reduce the decimal part of this product
to

to its value in the subsequent denomination, and so on — till there be either no decimal part remaining, or no lower denomination.—The integral parts of the several products form the answer. And the decimal part of the last product shews what fraction of the lowest denomination the given quantity contains besides.

Ex. 1. Required the value of .68125*l*.

$$\begin{array}{r}
 .68125\text{ }l. \\
 \underline{20} \\
 13.62500 \quad \text{value in shillings} \\
 \underline{12} \\
 7.500 \dots \text{value of .625s in pence} \\
 \underline{4} \\
 2.0 \text{ value of .5d. in farthings}
 \end{array}$$

Answer 13*s*. 7*d*. 2*f*.

Ex. 2. Required the value of .003125*l*.

$$\begin{array}{r}
 .003125 \text{ } l. \\
 \underline{20} \\
 0.062500 \text{ } s. \\
 \underline{12} \\
 0.7500 \dots d. \\
 \underline{4} \\
 3.00 \dots f. \quad \text{Answer 3 farthings}
 \end{array}$$

Ex. 3. Required the value of .0001 of a mile.

$$\begin{array}{r}
 .0001 \text{ miles} \\
 1760 \\
 \hline
 .1760 \text{ yards} \\
 3 \\
 \hline
 .528 \text{ feet} \\
 12 \\
 \hline
 6.336 \text{ inches.}
 \end{array}$$

Answer 6 inches

+ .336 of an inch

44. To reduce a quantity to a decimal of any denomination.

Rule. Reduce the quantity to a fraction of the proposed denomination, by art. 41. Reduce this fraction to a decimal.

Ex. 1. What decimal of a shilling is 3 pence?

$$3 \text{ pence} = \frac{3}{12} \text{ of a shilling; } \frac{3}{12} = .25, \text{ the answer.}$$

Ex. 2. What decimal of a mile is 6.336 inches?

$$6.336 \text{ inches} = \frac{6.336}{63360} \text{ of a mile; and } \frac{6.336}{63360} = .0001, \text{ the answer.}$$

If the given quantity contains more than one denomination, the easiest method of proceeding is, first to reduce that part, which is of the lowest denomination, to a decimal of the next superior, or 2d denomination,—then to reduce the sum of the decimal obtained, and the part, which is of the 2d denomination, to a decimal of the 3d denomination;—and so on.

Ex.

REDUCTION.

35

Ex. 3. What decimal of 1*l.* is 13*s.* 7*d.* 2*f.*?

$$2f. = \frac{2}{4}d. = .5d.; 7d. + 2f. = 7.5d. = \frac{7.5s.}{12}$$

$$= .625s.; 13s. + 7d. + 2f. = 13.625s. = \frac{13.625l.}{20}$$

$$= .68125l.$$

The operation may be performed as follows:

$$\begin{array}{r} 4 \overline{)2.} \\ 12 \overline{)7.5} \\ 20 \overline{)13.625} \\ \underline{.68125} \text{ Answer.} \end{array}$$

CHAPTER II.

OF ARITHMETICAL AND GEOMETRICAL SERIES.

ARITHMETICAL SERIES.

45. *Definition.* A SET of terms, increasing or decreasing by a common difference, is said to be in Arithmetical progression, and to form an Arithmetical Series.

Thus, 1, 3, 5, 7, 9, &c. are in Arithmetical progression, each term increasing by the common difference, 2. Again, 15, 12, 9, 6, form an Arithmetical Series, each term decreasing by the common difference 3. *In general*, if we denote the first term of a series by a , the common difference of the terms by d , and prefix to d , the sign +, or the sign -, according as the terms of the series increase or decrease, we shall have the expression $a, a \pm d, a \pm 2d, a \pm 3d$, &c. to denote any Arithmetical Series whatsoever.

46. *The n^{th} term of an Arithmetical Series, is equal to the first term, increased or diminished by $n-1$ times the common difference; according as it is an increasing or a decreasing series.*

For

For in any term of the *above series*, the coefficient of d is one less than the number denoting the place of the term; i.e. the n^{th} term is $a \pm n-1 \cdot d$.

47. *In an Arithmetical Series; the sum of the first and last terms is equal to the sum of any two intermediate terms, equally distant from the first and last, and to double the middle term, if there be a middle term, i. e. if the series consist of an odd number of terms.*

Thus, in the series, 1, 3, 5, 7, 9, 11; $1+11 = 3+9 = 5+7$; In the series 14, 11, 8, 5, 2; $14+2 = 11+5 = 8+8$. In general, as much as the 2d term exceeds or falls short of the 1st, so much the last but one falls short of or exceeds the last; as much as the 3d term exceeds or falls short of the 2d, so much the last but two falls short of, or exceeds the last but one; &c. &c.

48. *The sum of all the terms of an Arithmetical series, is equal to half the product, which arises from multiplying the sum of the first and last terms, into the number of terms. (Or, putting a to denote the first term; $a \pm d$ the second term, &c. l the last term; and consequently, $l \mp d$ the last but one; $l \mp 2d$, the last but two, &c. n the number of terms,*

and s their sum) $s = \frac{a+l \times n}{2}$.

For let the series be placed under itself in an inverted order, so that the last term may stand under the first—the last but one under the second,

cond, and so on. Then it will plainly appear either by inspection, or by art. 47, that, if each term be added to that above it, the sum of the two will constantly be $a+l$: hence since

$$\begin{aligned} a + a \pm d + a \pm 2d + a \pm 3d + \&c. &= s \\ \text{and } l + l \mp d + l \mp 2d + l \mp 3d + \&c. &= s \\ \hline a+l + a+l + a+l + \&c. \text{ n times} &= s+s \\ \text{or } \overline{a+l} \times n &= 2s; \therefore \text{dividing by 2,} \end{aligned}$$

$$s = \frac{\overline{a+l} + n}{2}$$

49. Since $l = a \pm \overline{n-1.d}$,

$$s = \left(\frac{a + a \pm \overline{n-1.d} \times n}{2} \right) = \overline{2a \pm \overline{n-1.d} \times \frac{n}{2}}$$

50. If b be put to denote the common difference, and be considered as positive or negative, according as the series is increasing or decreasing, in all cases we shall have $s = \overline{2a + \overline{n-1.b} \times \frac{n}{2}}$,

the sign $\mp$ alone being prefixt to $\overline{n-1.b}$.

51. If $a = b$, the series becomes the same as $b, 2b, 3b, 4b, \dots, mb$, where m is the number of terms; and $s = \left(\overline{a+l} \times \frac{m}{2} \right) = \overline{b + m.b} \times \frac{m}{2}$

$= \frac{\overline{m+1.b} \times m}{2}$. If v be prefixt to this series, it

becomes $0, b, 2b, \&c. \dots, mb$ (still under the Arithmetical form) having $\overline{m+1}$ terms; whose
sum

sum, according to the equation $s = \overline{a+l} \times \frac{n}{2}$, is
 $\overline{0+m} \times \frac{m+1}{2}$ or $\overline{m+1} \times \frac{b \times m}{2}$ as before.

52. Any *three* of the five quantities s, a, b, l, n , being given, the other two may be found from the two equations $l = a + \overline{n-1} \cdot b$, and $s = \overline{a+l} \times \frac{n}{2}$, or $s = 2a + \overline{n-1} \cdot b \times \frac{n}{2}$.

EXAMPLES.

1. Required the sum of 50 terms of the series 2, 4, 6, 8, &c.

Here $a = 2, b = 2; n = 50; \therefore s =$
 $(2a + \overline{n-1} \cdot b \times \frac{n}{2}) = 4 + 49 \times 2 \times \frac{50}{2} = 102 \times$
 $\frac{50}{2} = 2550.$

2. Required the sum of 5 terms of the series 11, 9, &c.

Here $a = 11; b = -2; n = 5; \therefore s = (22 + 4 \times -2 \times \frac{5}{2})$
 $= 14 \times \frac{5}{2} = 35.$

3. The first term is 14; the sum of 5 terms is 40. Required the series.

Here $a = 14; n = 5; s = 40. \therefore 40 = 28 + 4b \times \frac{5}{2}$

and

and $80 = \overline{28+4b} \times 5 = 140 + 20b$ hence
 $20b = 80 - 140 = -60$, and $b = -\frac{60}{20} = -3$.

$\therefore$ the series is 14, 14-3, &c. i. e. 14, 11, 8, 5, 2.

4. Required the sum of a decreasing series, whose first term is 35, common difference 3, and last term 20.

From the equation, $l = a + \overline{n-1} \cdot b$, we have in this case, $20 = 35 + \overline{n-1} \times -3$; hence

$$\overline{n-1} \cdot 3 = 35 - 20 = 15 \text{ and } \overline{n-1} = \frac{15}{3} = 5$$

$\therefore n = 6$;

$$\text{hence since } s = \overline{a+l} \times \frac{n}{2}, s = \overline{35+20} \times \frac{6}{2} = 165.$$

The number of terms in an increasing Arithmetical progression may be augmented at pleasure.

But in a decreasing progression, as many times as the common difference can be subtracted from the first term, so many terms, and so many only, can there be after the first. Thus, the progression 1, 2, 3, &c. admits of any number of terms, but the progression 19, 15, &c. admits of but five terms, viz. 19, 15, 11, 7, 3.

This limitation takes place in all those decreasing Arithmetical progressions, which may be supposed to have any direct signification,—or to have been derived in a direct manner from any existing case.

But

But if we extend the Algebraic operations to abstract negative quantities, it is plain that by following the same law of continually subtracting the common difference, *any* Arithmetical progression may be continued at pleasure. Thus in the above series 19, 15, 11, 7, 3; we may still continue the progression by (algebraically) subtracting 4; and we shall have for the next term $3-4$, or -1 ; for the next succeeding, $-1-4$, or -5 , and so on. And, agreeably to the same method of employing the symbols, the first term itself may be the negative. e. g. -1 , -2 , -3 , may be considered as a decreasing Arithmetical series, whose common difference is 1. And it is obvious from the method in art. 48 that the *Algebraic* sum of such series will still be had from the equation

$$s = 2a + n-1. b \times \frac{n}{2}.$$

5. How many terms of the series 19, 18, 17, &c. added together, amount to 124?

Here $s = 124$; $a = 19$; $b = -1$

$$\text{hence } 124 = 38 \times \frac{n-1}{2} + -1 \times \frac{n}{2}$$

$$\text{and } 248 = 38n - n^2 + n$$

$$n^2 - 39n = -248$$

$$n^2 - 39n + \left(\frac{39}{2}\right)^2 = -248 + \left(\frac{39}{2}\right)^2 = \frac{529}{4}$$

$$\therefore n - \frac{39}{2} = \pm \sqrt{\frac{529}{4}} = \pm \frac{23}{2}$$

F

and

$$\text{and } n = \frac{39 \pm 23}{2} = 31 \text{ or } 8.$$

$\therefore$ 8 is the direct answer, or the 8 terms 19, 18, 17, 16, 15, 14, 13, 12, amount to 124; but if to these 8 be added the succeeding 23, viz. 11, 10 2, 1, 0, - 1, - 2 - 10, - 11, the sum of the whole 40 will still be the same.

53. In assuming three of the five quantities, s, a, b, n, l , we are at liberty (n , a *whole number*, being one of the three) to take any quantities whatsoever for the other *two*, since from any assumption there will result some value or other for the quantities sought. But when n is made the unknown quantity, three of the other four cannot be assumed *ad libitum*; for unless they be such as to give n a *whole number*, the equations

$$l = a + n - 1, \quad b, \quad \text{and} \quad s = 2a + n - 1, \quad b \times \frac{n}{2},$$

have no relation to an Arithmetical series, since any series must have a *real whole number* of terms. It is very easy to see, why, from the nature of such series, the assumption of two out of three of the quantities s, a, b, l , restricts the third to certain values. Thus, if a and b be assumed, it is obvious that such assumption determines each term of the series, and consequently determines the sum of *one, two, three*, or of *any other number* of terms; i. e. restricts s to certain values, other than which it may not be. And if s be assumed of any intermediate value between the first

first term and the sum of two, sum of two and sum of three, &c, the resulting value of n will not be integral.*

If one value of n found from the equation $s = 2a + n-1. b \times \frac{n}{2}$, be an integral negative

number, $-p$, it indicates that the sum of p terms of the series $a, a+b, \&c.$ taken the contrary way, with their signs changed, making $a-b$ the first term, will agree with the proposed sum. For

since $s = 2a + n-1. b \times \frac{n}{2}$, and $n = -p$,

$$s = 2a - p-1. b \times -\frac{p}{2} = -2a + p+1. b \times \frac{p}{2}$$

$$= 2 \times b-a + p-1. b \times \frac{p}{2}, \text{ which is the sum of a}$$

series, whose first term is $b-a$, common difference b , and number of terms p ; i. e. the sum of p terms of the series $b-a, 2b-a, \&c.$ or $a-b, a-2b, \&c.$ taken negatively.

* It is not to be inferred that the value of n will be a corresponding fraction, denoting that the sum of so many terms, and such a part of the next, will agree with the proposed sum, — Such an inference is not warranted by the derivation of the equation

$$s = 2a + n-1. b \times \frac{n}{2}.$$

Ex. 1; $a = 4$; $b = 3$; $s = 14$; required n ,

$$14 = 8 + \overline{n-1} \cdot 3 \times \frac{n}{2}$$

$$28 = 8n + 3n^2 - 3n$$

$$n^2 + \frac{5n}{3} = \frac{28}{3}$$

$$\therefore n + \frac{5}{6} = \pm \sqrt{\frac{28}{3} + \frac{25}{36}} = \pm \sqrt{\frac{361}{36}} = \pm \frac{19}{6}$$

$$\text{and } n = -\frac{5}{6} \pm \frac{19}{6} = \frac{8}{3} \text{ or } -4$$

$\therefore$ four terms of the given series, taken the contrary way, and with their signs changed, viz. 1, -2, -5, -8, with their signs changed, amount to 14.

Ex. 2. How many terms of the series 10, 14, &c. amount to 0?

$$s = 0 = 20 + \overline{n-1} \times 4 \times \frac{n}{2} \therefore \text{dividing by } \frac{n}{2}$$

$$20 + \overline{n-1} \times 4 = 0 \text{ whence } n = -4.$$

It may be useful to the reader to propose to himself any four of the five s , a , b , n , l , and endeavour to find one of them in terms of the other three, from the two general equations.

GEOMETRICAL SERIES.

54. Definition, A set of terms, increasing or decreasing in such a manner, that each is the same multiple, part or parts of the preceding, or, in other words, in such a manner, that the same ratio as the 1st term has to the 2d, the 2d has to the 3d, and so on, to the end of the series, is said to be in Geometrical Progression, and to form a Geometrical Series. Thus 2, 6, 18, 54, &c. is a Geometrical Progression, each term being triple the preceding term. Again, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ is a Geometrical Series, each term being half the preceding.

55. Any Geometrical Series may be represented by the form $a, ar, ar^2, ar^3, \&c.$ where each term is equal to the preceding term multiplied into r . For if the 1st term of a geometrical series be a , and the 2d be b , since $b = a \times \frac{b}{a}$, calling $\frac{b}{a}$ r , the 2d term will be ar ; and since $a : ar :: ar : ar^2 :: ar^2 : ar^3 : \&c.$ the succeeding terms will, by the definition, be $ar^2, ar^3, \&c.$

56. This constant multiplier r , or that quantity, which, multiplied into the 1st term, produces the 2d, is called the common ratio, and is
obviously

obviously always equal to $\frac{2d \text{ term}}{1st \text{ term}}$

or to $\frac{\text{any term}}{\text{preceding term}}$.

57. The n^{th} term of a Geometrical series is equal to the $\overbrace{\text{first term} \times \text{common ratio}}^{n-1}$.

For in any term of the series $a, ar, ar^2, ar^3, ar^4, \&c.$ the index of r is one less than the number denoting the place of the term; i. e. the n^{th} term is ar^{n-1} .

58. The product of the extreme terms in a geometrical series is equal to the product of any two terms equally distant from the extremes, and, if the number of terms be odd, to the square of the middle term.

Thus in the series 2, 4, 8, 16, 32, 64,
 $2 \times 64 = 4 \times 32 = 8 \times 16$; in the series

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \frac{1}{162}, \quad \frac{1}{2} \times \frac{1}{162} =$$

$$\frac{1}{6} \times \frac{1}{54} = \frac{1}{18} \times \frac{1}{18} \quad \text{In general, let } a \text{ be the first}$$

term, r the common ratio, and y the last term;
 and, consequently, $\frac{y}{r}$ the last but one; $\frac{y}{r^2}$ the last

but two; &c. then the series will be $a, ar, ar^2, \&c.$

$$\dots, \frac{y}{r^2}, \frac{y}{r}, y; \text{ and it is evident that } a \times y =$$

$$ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2} = \&c.$$

59. The

59. *The sum of all the terms of a geometrical series is equal to the quotient, which arises from dividing the difference between the first term and last term $\times$ common ratio, by the difference between unity and the common ratio; or, using the above notation, and putting s for the sum, $s = \frac{yr - a}{r - 1}$.*

For if $a + ar + ar^2 + \dots + \frac{y}{r} + \frac{y}{r^2} + y = s$,

multiplying both sides by r

$$ar + ar^2 + ar^3 + \dots + \frac{y}{r} + y + yr = rs$$

and subtracting one * equation from the other

$$a - yr = s - rs = s \times \overline{r - 1}$$

$$\therefore s = \frac{a - yr}{r - 1}.$$

60. *The sum of all the odd terms (1st, 3d, &c.) of a geometrical series, diminished by the sum of all the even terms (2d, 4th, &c.) is equal to*

* If r be greater than unity, the value of the sides of the upper equation is less than the value of the sides of the lower, and the upper equation is to be subtracted from the lower; if r be less than unity, the contrary happens, and the lower equation is to be subtracted from the upper. In the former case $s = \frac{yr - a}{r - 1}$; in the latter,

$$s = \frac{a - yr}{1 - r}.$$

first

first term + last term $\times$ common ratio, or to
common ratio + 1

first term — last term $\times$ common ratio,
common ratio + 1

according as the number of terms is odd or even. Or

$$s = \frac{a \pm yr}{r + 1}.$$

For if $a - ar + ar^2 \dots \mp \frac{y}{r} \pm y^* = s$

multiplying by r

$$\begin{array}{r} ar - ar^2 + ar^3 \dots \mp y \mp yr = rs \\ \hline \text{by addi. } a \pm yr \qquad \qquad \qquad = s + rs = s \times r + 1 \end{array}$$

$$\therefore s = \frac{a \pm yr}{r + 1}.$$

61. The sum of all the even terms, diminished by the sum of all the odd terms, is equal to

$$\frac{\text{last term} \times \text{common ratio} - \text{first term}}{\text{common ratio} + 1},$$

$$\text{or } s = \frac{yr - a}{r + 1}.$$

* If the number of terms be odd, the sign of this last term is +; if even it is —.

For

For if $-a + ar - ar^2 + \dots - \frac{y}{r} + *y = s$

$$-ar + ar^2 - ar^3 + \dots - y + yr = rs$$

and by addⁿ. $-a + yr = s + rs = s \times r + 1$

$$\text{and } s = \frac{yr - a}{r + 1}.$$

62. By putting r to denote that quantity, be it positive, or negative, which, multiplied into any term, produces the next succeeding with its proper sign, and by putting a and y to denote the first and last terms with their proper signs prefixt, any geometrical series whatsoever, whether its terms be all positive, or alternately positive and negative, is made to agree with the form

$$a + ar + ar^2 + ar^3 \dots + \frac{y}{r} + y.$$

Thus in the series $2, -4, +8, -16, +32$; putting $r = -2$, we shall have $-4 = 2 \times r$; $8 = -4 \times r$, or $= 2 \times r^2$; $-16 = 8 \times r$, or $= 2 \times r^3$; $32 = -16 \times r$, or $= 2 \times r^4$. In the series $-2, +6, -18, +54$; putting $a = -2$, and $r = -3$, we shall have $6 = -2 \times r$; $-18 = 6 \times r$; or $= 2 \times r^3$; &c. And the expressions for s , found in the three last articles, will all

* As the enquiry is here restricted to real quantities, the number of terms in *this* series *must* be even; else the sum of the negative terms would exceed the sum of the positive terms.

coalesce in one, viz. $s = \frac{yr-a}{r-1}$. For if

$$a + ar + ar^2 \dots + \frac{y}{r} + y = s,$$

$$ar + ar^2 + ar^3 \dots + y + yr = rs$$

and subtracting the upper equation from the lower,

$$yr-a = rs - s = s \times \overline{r-1}$$

$$\text{and } s = \frac{yr-a}{r-1} *$$

If the sides of the upper equation be of greater value than the sides of the lower, $yr-a$ is negative; but at the same time $rs-s$ will be negative, so that the *algebraical* value of $\frac{yr-a}{r-1}$ is still the true value of s .

63. Since y , the n^{th} . term of the series, is equal to ar^{n-1} (art. 57) $s = \frac{ar^{n-1} \times r - a}{r-1} = \frac{ar^n - a}{r-1}$.

* Perhaps the following method of applying this demonstration, may seem less artificial.

$$a + ar + ar^2 \dots + \frac{y}{r} + y = s$$

$$\therefore ar + ar^2 \dots + \frac{y}{r} + y = s - a;$$

$$\text{but } ar + ar^2 \dots + \frac{y}{r} + y = a + ar + ar^2 \dots + \frac{y}{r^2} + \frac{y}{r} \times r \\ = \overline{s-y} \cdot r \therefore \overline{s-y} \cdot r = s - a$$

$$\text{hence } rs - s = yr - a, \text{ and } s = \frac{yr-a}{r-1}.$$

64. Any

64. Any three of the five quantities, s, a, r, y, n being given, the other two may be found from the above equations,

EXAMPLES.

1. Required the sum of 15 terms of the series
1, 2, 4, &c.

Here $a = 1$; $r = 2$; $n = 15$;

$$\therefore s = \frac{1 \times 2^{15} - 1}{2 - 1} = 2^{15} - 1 = 32767.$$

2. Required the sum of 6 terms of the series

21, $-3, +\frac{3}{7}$ &c.

$$r = -\frac{1}{7}, \therefore s = \frac{21 \times -\frac{1}{7} \left(1 - \left(-\frac{1}{7} \right)^6 \right) - 21}{-\frac{1}{7} - 1} = \frac{21 \times \frac{1}{7^6} - 21}{-\frac{1}{7} - 1}$$

$$= \frac{21 - 21 \times \frac{1}{7^6}}{\frac{1}{7} + 1} = \frac{21 \times 7^6 - 21}{7^6} \div \frac{8}{7}$$

$$= \frac{21 \times 7^6 - 1}{8 \times 7^5} = \frac{3 \times 7^6 - 1}{8 \times 7^5} = \frac{3 \times 117648}{8 \times 2401}$$

$$= \frac{3 \times 14706}{2401}$$

3. The 5th term is 625, and the common ratio is -5 ; required the series.

$y = 625$; $r = -5$; $n = 5$; $\therefore$ from the equation

$$y = ar^{n-1}, 625 = a \times -5^4 = a \times 5^4$$

$$\therefore a = \frac{625}{5^4} = 1; \text{ and the series is}$$

$$1 - 5 + 25 - 125 + 625.$$

The methods given in chapter 7, Vol. I. of deducing an equation, in which the unknown quantity occupies one side, and the known quantities the other, cannot be exactly applied here, when n is the quantity sought; for from the equation

$$s = \frac{ar^n - a}{r - 1} \text{ we can only deduce that } r^n = \frac{s r - s + a}{a}$$

and cannot by means of the symbols employed in Algebra, viz. the symbols for *addition*, *subtraction*, *multiplication*, *division*, *involution*, and *evolution*, state the value of the index n in terms of s , r , and a .

Yet is n as much determined from that equation, as if its value could be so stated; for there can be *but one* power of r , which is equal to

$$\frac{s r - s + a}{a}, \text{ and that power may be found in any}$$

particular case, by trial, or by previous knowledge of the different powers of r . And in fact, it is thus that the values of quantities are found, when division and evolution are concerned.

Thus,

Thus, if we have given that $x = \frac{56}{7}$; in order to know what x really is, we must either by trial discover, or have it previously established in our minds, how many *sevens* amount to 56. The only difference in the two cases is that in division and evolution, the system of notation, by enabling us to discover the parts of the answer separately, does so limit the number of trials, as to render the operation with large numbers nearly as easy as with small. Thus, in finding the square root of 625, it is not necessary to enquire at once (by means of successive trials, or of previous experience) what number taken as often as there are units in it, amounts to 625 — the operation is abridged by first enquiring what number multiplied into itself amounts to, or at least as nearly as possible amounts to 6; &c. But this artifice we cannot so easily avail ourselves of, in enquiring what power a given number must be raised to, in order that the amount may be a certain proposed number.

In these equations, where the unknown quantity is an index, such as the equation $c^x = d$, we may say that x is *that number, to the power of which if c be raised, the result will be d* ; and if we had symbols to express that phrase, the value of the unknown quantity would seem as clearly determined as in other cases. Or, because $(c^x : 1) = (d : 1)$, and $(c^x : 1) = x \times (c : 1)$,
(see

(see ratios) we may say, that $x \times (c : 1) = (d : 1)$, and $x = \frac{d : 1}{c : 1}$; and since d and c are, by hypothesis, known, it may be supposed that it is known how often the ratio $c : 1$ is contained in the ratio $d : 1$.

When r is the quantity sought, and s , a and n are given, and n is greater than 2, the equation to the unknown quantity, $ar^n - sr = a - s$, or $r^n - \frac{s}{a} \times r = \frac{a-s}{a}$, is of higher dimensions than a quadratic, and consequently cannot be solved by the methods in chapters 7 and 8, Vol. I. Nor, indeed, has any general expression for the value of the unknown quantity in equations of more than four dimensions, been hitherto discovered. Yet that value may truly be said to be *known*, because it is *determined* by the equation, and may be found by trial. E. g. let $x^5 - 3x = 234$; it is plain that it is not every number, whose 5th power, diminished by three times its 1st power, leaves 234; though, for ought that appears, there may be more than one number, which will answer that condition, for we know that in the quadratic equation, $x^2 - px = -r$, x has two values. If we substitute a very large number for x , it is evident that $x^5 - 3x$ will far exceed 234; if we substitute 10, it will exceed 234; if we substitute 2, it will fall short of it; $\therefore$ we know that the true value of x is between 2 and 10; and upon trial we find it to be 3.

Ex.

Ex. 4. How many terms of the series $1, -6, +36, -\&c.$ amount to 1111?

$$a = 1; r = -6; s = 1111$$

$$\therefore 1111 = \frac{1 \times \overline{-6}^n - 1}{-6 - 1} = \frac{\overline{-6}^n - 1}{-7}$$

and $1111 \times -7 = \overline{-6}^n - 1$; hence

$1 - 7 \times 1111 = \overline{-6}^n = -6^n$, and $7 \times 1111 - 1$ or $7776 = 6^n$; but the 5th power of 6 is 7776,

$$\therefore n = 5.$$

By solving this example from the equation $s = \frac{a \pm ar^n}{r \pm 1}$, (see art. 60) the powers of negative quantities are avoided. For then

$$1111 = \frac{1 + 1 \times 6^n}{6 + 1} = \frac{1 + 6^n}{7}$$

$\therefore 1111 \times 7 = 1 + 6^n$, and $1111 \times 7 - 1 = 6^n$ as before; $\therefore \&c.$

65. In an increasing geometrical progression, or one, in which r exceeds unity, by augmenting the number of terms, their sum may be increased sine limite. Thus no number, howsoever great, can be proposed, but may be exceeded by adding together the terms of the progression $2 + 4 + 8 + \&c.$

* The n^{th} power of -6 ($\overline{-6}^n$) is positive or negative, according as n is even or odd; if it be negative, it is the same as -6^n ; and that it is negative in this case we know, because $1 - 7 \times 1111$ is negative. But that n is an odd number appears also from the sum of the series being positive; since the sum of an even number of terms in an increasing geometrical series, whose terms are alternately $+$ and $-$, (beginning with $+$) must be negative.

continued

continued at pleasure. But in a decreasing geometrical progression, or one, in which r is less than unity, the sum of the terms, howsoever far they be extended, can never reach a certain limit. Thus howsoever many terms be taken in the series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$ their sum can never amount to 2. (See division, art. 170, Ex. 5.)

In general, since $s = \frac{ar^n - a}{r - 1}$, r^n , and consequently ar^n , may be increased* at pleasure by increasing n , and consequently the value of the whole expression $\frac{ar^n - a}{r - 1}$ may be increased *sine limite*. But when r is less than unity, by increasing n , r^n and consequently ar^n , decreases *sine limite*, so that $\frac{ar^n - a}{r - 1}$, or rather $\frac{a - ar^n}{1 - r}$, has for the limit of its

* Suppose $r = \frac{p+q}{p}$; then $r^n = \frac{(p+q)^n}{p^n} = \frac{p^n + np^{n-1}q + \&c.}{p^n}$
 $= 1 + \frac{nq}{p} + \&c.$ Now whatever p and q be, $\frac{nq}{p}$ may, by increasing n , be increased *sine limite*.

Again, suppose $r = \frac{p}{p+q}$; then $r^n = \frac{p^n}{(p+q)^n}$, which is the reciprocal of $\frac{(p+q)^n}{p^n}$; and since no number can be found which $\frac{(p+q)^n}{p^n}$ may not exceed, no fraction can be assigned so small, but $1 \div \frac{(p+q)^n}{p^n}$ may be still smaller.

value $\frac{a}{1-r}$; i. e. the sum of no number of terms whatsoever can be so great as $\frac{a}{1-r}$, but always falls short of it by $\frac{ar^n}{1-r}$; yet since $\frac{ar^n}{1-r}$ is diminishable at pleasure, nothing short of $\frac{a}{1-r}$ can be assigned, which, by continuing the progression, the sum of its terms may not exceed.

This limit $\left(\frac{a}{1-r}\right)$ of the values of s is usually called the sum of the series continued *ad infinitum*.

Ex. 1. Required the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \dots$ *ad infinitum*.

Here $a = 1$, and $r = \frac{1}{2}$;

$$\therefore s = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{2-1}{2}} = \frac{2}{2-1} = 2.$$

Ex. 2. Required the sum of the series $\frac{4}{5} + \frac{4}{5} \times \frac{5}{6} + \frac{4}{5} \times \frac{5^2}{6^2} + \dots$ or $\frac{4}{5} + \frac{2}{3} + \frac{10}{18} + \dots$ *ad infinitum*.

H

Here

$$\begin{aligned} \text{Here } a &= \frac{4}{5}, \text{ and } r = \frac{5}{6}; \therefore s = \frac{4}{5} \div \overline{1 - \frac{5}{6}} \\ &= \frac{4}{5} \div \frac{1}{6} = \frac{24}{5}. \end{aligned}$$

Ex. 3. Required the sum of the series $\frac{1}{2} - \frac{1}{2 \cdot 3}$
 $+ \frac{1}{2 \cdot 3^2}$ &c. *ad infinitum*.

$$\begin{aligned} \text{Here } a &= \frac{1}{2}, \text{ and } r = -\frac{1}{3}; \therefore s = \frac{1}{2} \div \overline{1 + \frac{1}{3}} \\ &= \frac{1}{2} \div \frac{4}{3} = \frac{3}{8}. \end{aligned}$$

Any number, consisting of the same figure or figures repeated, forms a geometrical series.

Thus 66666 is a geometrical series of 5 terms, whose 1st term is 6 or 6000, and common ratio 10 or $\frac{1}{10}$, according as we suppose the series to begin at the 6 *units* or at the 6 *tens of thousands*: Again, 2727.27272727 is a geometrical series of 6 terms, where $a = \frac{27}{100,000,000}$ or 2700, and $r = 100$ or $\frac{1}{100}$.

Hence, from the equation $s = \frac{a}{1-r}$, we may find the limiting value of any *circulating decimal*, or the vulgar fraction, from which it arose. (see Dec. fractions, art. 160, 161.)

Ex.

Ex. 1. Required the value of .333, &c. *ad infinitum*. $a = \frac{3}{10}$; $r = \frac{1}{10}$; $\therefore s = \frac{3 \div 10}{1 - 1 \div 10}$
 $= \frac{3}{10 - 1} = \frac{1}{3}.$

Ex. 2. Required the value of .0909, &c. *ad infinitum*. $a = \frac{9}{100}$; $r = \frac{1}{100}$; $\therefore s = \frac{9 \div 100}{1 - 1 \div 100}$
 $= \frac{9}{100 - 1} = \frac{1}{11}.$

The sum of a series *ad infinitum* is, in practice, more expeditiously obtained by pursuing the method in art. 62, than by employing the expression

$\frac{a}{1-r}$, thence deduced. *E. g.* if $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

+ &c. = s , multiplying both sides by 2,

$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$ (which is $2 + s$)

= $2s$ $\therefore 2 = 2s - s = s$. If .090909, &c. = s ,

09.0909, &c. = $100s$, or $9 + s = 100s$; $\therefore 9$

= $99s$, and $s = \frac{1}{11}.$

If 123.123123, &c. = s , 123123.123123, &c.
 = $1000s$ and $123000 = 999s$ whence $s = \frac{123000}{999}.$

To find the sum of the series

$a + ar + ar^2 + \dots + ar^{n-1}$ is the same thing as to find what fraction, when the numerator is actually divided by the denominator, will produce that series of terms; and such fraction is found to

be $\frac{ar^n - a}{r - 1}$. And to find the limiting value of

$a + ar + ar^2 + \&c. \text{ ad infinitum}$ is to find what fraction will by the same actual division produce the series $a + ar + ar^2, \&c.$ with a perpetual remainder $\rightarrow ar^n$, which fraction is $\frac{a}{1-r}$ (see division Ex. art. 170.)

CHAPTER III,

OF INCOMMENSURABLES.

68. **M**ATTER or extension, when it becomes the subject of Arithmetical computation, is considered as consisting of congeries of similar parts.—Lines as being composed of a number of lesser lines; surfaces of lesser surfaces; and solids of lesser solids. And an enquiry is instituted concerning how often one portion of extension is contained in another of the same kind; or, supposing the two lines, surfaces, or solids to be divided into any lesser portions, all of equal magnitude, concerning what number of these portions will be found in each of them. And if any two portions of linear extension, that can be constructed, could be formed by putting together in one line certain number of lesser particles of linear extension, all equal among themselves,—so many to make up one of the portions, and so many to make up the other; if in the same manner any two portions of superficial extension could be formed by joining the bounding lines of equal superficial particles; and of solid extension by the junction of equal solid particles, then all Mathematical

mathematical quantities would be commensurable: For whatever is made quantity, is measured either by number, or by extension; the only ideas we have of magnitude, being of different numbers of things, or of things of different dimensions.

69. But the different portions of extension, which occur in Mathematical enquiries, are not always what could result from any such composition by addition of equal parts. Two lines, two surfaces, or two solids may be constructed, one of which shall be no exact multiple, part, or parts whatsoever of the other. Thus (as will be shewn) the side of the square is incommensurable with the diagonal; it is not $\frac{n}{m}$ ths of the diagonal, what-

ever m and n be taken:—If the diagonal be divided into a million equal parts, the side will not be found exactly equal to any certain number of such parts; nor if the diagonal be divided into any other number of equal parts, how great soever that number be, and consequently, how minute soever the parts be, will the side be exactly equal to a certain number of them.

70. Magnitudes of this description are said to be incommensurable with each other, as having no common measure.

71. Numbers, as was observed in Chapter I. Vol. I. are the exponents of magnitudes: To say that any two numbers are commensurable with each

each other, is only to say that two magnitudes, one of which can be expounded in terms of the other, are commensurable—an identical proposition.*

72. To magnitudes, that are incommensurable, it is evident that the doctrine of ratios, as delivered in Chap. IX, Vol. I. cannot be applied. For the definition there given of equal ratios, is restricted to such magnitudes, as are multiples or parts one of the other; or, which is the same thing, to such as can be expressed by numbers; and will not determine whether the ratio between two incommensurable magnitudes be, or be not equal to that between other two magnitudes; or, to speak more strictly, proportionality would not from that definition attach to incommensurables.

73. Yet, since there is found between certain incommensurables a relation very analogous to that defined in the article quoted above—a re-

* It may not be amiss to observe here, that this commensurability of numbers extends to fractions, since any two fractions must be multiples or parts one of the other; or in other words no ratio is expressible in fractions but what is also expressible in whole numbers.

For let $\frac{a}{b}$ and $\frac{c}{d}$ denote any two fractions, a, b, c and d , being whole

numbers; then $\frac{c}{d} = \frac{a}{b} \times \frac{bc}{ad}$, or is that multiple part or parts of

$\frac{a}{b}$ which is expressed by the fraction $\frac{bc}{ad}$: In other words $\frac{a}{b} : \frac{c}{d}$

$:: ad : bc$.

lation

lation, which the mind in every view of the subject does as it were naturally consider as belonging to that proportionality, and which, in quantities that are commensurable, does altogether coincide with it,—it has been found serviceable to Mathematics to lay down a more general definition of proportionality; one that shall include quantities incommensurable as well as commensurable.

In order to prepare the reader to admit the utility of this definition, and more clearly to discern its bearings and connexion with that already given, the following account may be premised.

74. It is found that there may be four magnitudes, A, B, C, D , either all of the same kind, or A and B of one kind, and C and D of another, of such a nature, that though A is incommensurable with B , and C with D , yet in whatever manner A exceeds or falls short of B , in the same manner C exceeds or falls short of D : so that, for example, if A be greater than B , C is greater than D ,—if A be greater than the double of B , C is greater than the double of D ,—if A be less than the triple of B , C is less than the triple of D ;—

In general, if A be greater than $\frac{m}{n}$ ths of B , C is greater than $\frac{m}{n}$ ths of D ,—if A be less than

$\frac{m+1}{n}$ ths of B , C is less than $\frac{m+1}{n}$ ths of D . So

that whatever multiple, part, or parts of B differs very little from A , the same multiple, part, or parts of D differs very little from C ; and if A differs by *excess*, C differs by *excess*,—if by *defect*, C by *defect*. An example of this is found in the sides and diagonals of any two squares: let S and D denote the side and diagonal of any square, s and d , the side and diagonal of any other square. Then D is greater than S ; also d is greater than s .— D is less than twice S ; also d is less than twice s .— D is greater than 14 10ths of S , but less than 15 10ths; also d is greater than 14, but less than 15 10ths of s .— D is greater than 141 100ths of S , but less than 142 100ths of S ; also d is greater than 141, but less than 142 100ths of s .— D is greater than 1414, but less than 1415 1000ths of S ; also d is greater than 1414, but less than 1415 1000ths of s : And thus we may proceed for ever, diminishing **sine limite* the difference between the defective and the excessive values of D or d ,—the limits of d in terms of s being always found the same as the

* For the difference between 14 and 15 10ths of S , is $\frac{S}{10}$; between 141 and 142 100ths of S , $\frac{S}{100}$; and the succeeding differences are but a 1000th part, a 10000th part, an 100,000th part, &c. respectively, of S .

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limits

limits of D in terms of S ,—yet never finding a fraction, that will exactly express D and d in terms of S and s ; and that, whether these decimals, or *any other* fractions be employed.*

75. Any four magnitudes of this nature are made proportional by the following

DEFINITION.

The ratio A to B is the same as the ratio of C to D , if, taking m and n any two numbers whatsoever, when A is less than $\frac{m}{n} B$, C is less than $\frac{m}{n} D$; when A is equal to $\frac{m}{n} B$, C is equal to $\frac{m}{n} D$; when A is greater than $\frac{m}{n} B$, C is greater than $\frac{m}{n} D$.

This is Euclid's definition expressed fraction-wise, according to the modern method of Algebraic notation. For if A be less than $\frac{m}{n} B$, and C less than $\frac{m}{n} D$, nA is less than mB , and nC is less than mD . If A be equal to $\frac{m}{n} B$, and C equal to

* It is *assumed here*, that D and d are thus inexplicable in terms of S and s .

$\frac{m}{n} D$, nA is equal to mB , and nC equal to mD :

If A be greater than $\frac{m}{n} B$, and C be greater than

$\frac{m}{n} D$, nA is greater than mB , and nC greater than

mD . Or, "any equimultiples whatsoever, (nA , nC) of the first and third being taken, and any equimultiples whatsoever (mB , mD) of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth." Simson's Euclid. B. 5. definition 5.

76. If commensurable quantities be proportional according to this definition, they are proportionable according to the former definition.

For if A be commensurable with B , and C with D , A is equal to certain parts of B , and C to certain parts of D ; let $A = \frac{m}{n} B$, then if A , B , C , D be proportional, $C = \frac{m}{n} D$; but if $A = \frac{m}{n} B$, and

$C = \frac{m}{n} D$, $A : B :: C : D$ according to the former definition.

77. If four magnitudes be proportional according to the former definition, they are proportional according to this definition.

For let $A : B :: C : D$ according to the former definition;—let $A = \frac{p}{q} B$, and consequently $C = \frac{p}{q} D$. Take m and n any two numbers whatsoever. Then if A be less than $\frac{m}{n} B$, ($\frac{m}{n}$ must be greater than $\frac{p}{q}$, and therefore, since $C = \frac{p}{q} D$) C is less than $\frac{m}{n} D$; if A be equal to $\frac{m}{n} B$, ($\frac{m}{n}$ must be equal to $\frac{p}{q}$, and) C is equal to $\frac{m}{n} D$; if A be greater than $\frac{m}{n} B$, ($\frac{m}{n}$ must be less than $\frac{p}{q}$, and) C is greater than $\frac{m}{n} D$: Agreeably to the definition.

78. If there be three magnitudes A, B, C , and three numbers, a, b, c , whole or fractional, and A be to B as a to b , also B to C as b to c , A will be to C as a to c :—And if A, B, C be in continued proportion so are a, b, c .

For since $A : B :: a : b$, $A = \frac{a}{b} \times B$ and
since

since $B : C :: b : c$, $B = \frac{b}{c} \times C$

$$\therefore A = \frac{a}{b} \times \frac{b}{c} \times C = \frac{a}{c} \times C$$

and $A : C :: a : c$.

also if $A : B :: B : C$, then since

$$A : B :: a : b, \quad a : b :: B : C$$

but $B : C :: b : c \therefore a : b :: b : c$.

79. If there be *any number* of magnitudes A, B, C, D , &c. and *as many numbers* a, b, c, d , &c. and we have these proportions, viz.

$A : B :: a : b$; $B : C :: b : c$; $C : D :: c : d$; &c. then will the ratio between any two of the magnitudes A, B, C, D , &c. be equal to the ratio between the corresponding two of the numbers a, b, c, d , &c.—And if A, B, C, D , &c. be in continued proportion, so will a, b, c, d , &c.

$$\text{For } D = \frac{d}{c} C = (\text{by last art.}) \frac{d}{c} \times \frac{c}{b} A =$$

$\frac{d}{a} A \therefore A : D :: a : d$; and in the same man-

ner may it be proved of any other two.

Also if $A : B :: B : C :: C : D :: \&c.$ then, since, by the last art. $a : b :: b : c$, and $b : c :: c : d$ and so on, $a : b :: b : c :: c : d :: \&c.$

80. The real existence of *incommensurables* may be demonstrated by proving, that if an integral number has no integral root, neither has it any fractional root, e. g. by proving, that there are no two numbers, a and b , such, that $1 + \frac{a}{b}^2 = 2$. For

it is shewn, in Euclid (B. 6, P. XIII,) that a mean proportional between any two given lines may be found by construction: Let two lines be in the ratio of 1 to p , (p being an integral number) and be denoted by A and pA ; and let M denote the mean proportional between A and pA ; then if all lines were commensurable with each other, M must necessarily be commensurable with A ; —

let $M = \frac{m}{n} \times A$; then, by the definition of proportionality, $pA = \frac{m}{n} \times M = \frac{m}{n} \times \frac{m}{n} A = \frac{m^2}{n^2}$

$\times A$, and $p = \frac{m^2}{n^2}$ whence $\sqrt{p} = \frac{m}{n}$. i. e. if all

lines were commensurable, any number p would have a square root; if therefore it can be shewn that there are numbers, which cannot be produced by the multiplication of any fraction whatsoever into itself, it will be proved that incommensurables do exist.

Now if it were true that whatever number p be taken to denote, $\sqrt{p}$ may equal $\frac{m}{n}$, or p equal $\frac{m^2}{n^2}$,

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we should have $p n = \frac{m^2}{n}$; i. e. although (by hypothesis) m itself be not divisible by n without a remainder, yet $m \times m$ would be so divisible, containing n , $p n$ times; or n would be a factor of $m \times m$: But it may be proved that if two or more numbers be multiplied together, (whether they be the same or different numbers) the product will contain no factor but what multipliers contain; i. e. if the product be divisible without a remainder by any number, that number is either one of the multipliers, or a factor of one of the multipliers, or the product of two or more of the multipliers, or their factors. Thus if we multiply together the numbers 2, 6, 7, 11, the product, 924 will be divisible by no number except 2, 2, 3, 7, 11 and the products that may be formed out of them such as 2×2 , 2×3 , 7×3 , &c.

81. In order therefore to establish the existence of incommensurables, it is necessary to consider the nature and composition of integral numbers with respect to their factors.

OF THE COMPOSITION OF NUMBERS.

Note ; in the succeeding articles by "a number" is meant any integer,—and among the factors of a number is not included either the *number itself*, or *unity*; thus though $a = a \times 1$, a and 1 are not here called factors of a .—Also, when any number

ber is said to be divisible by another, it is meant to be divisible by it without a remainder. In this sense 6 is divisible by 3, but not by 4.

82. EVERY number is either prime or composite.

83. A prime number is one, that is not divisible by any other number, except by unity; or in other words, that is not composed by the repetition of 2, of 3, of 4, or of any higher number.

Thus 1, 2, 3, 5, 7, 11, 13, 17 are prime numbers.

84. A composite number is one, that is divisible by some other number greater than unity; or, that is composed by the repetition of 2, of 3, of 4, or of some higher number.

Thus 4 is a composite number, being divisible by 2, or equal to $2+2$; again, 6 is a composite number, being divisible by 2 or by 3; or equal to $3+3$ or to $2+2+2$.

85. Every composite number is the product of two or more prime numbers, greater than unity; which are called its prime factors.

Thus the prime factors of 30 are 2, 3, 5, 30 being equal to $2 \times 3 \times 5$; the prime factors of 175 are 5, 5, 7,—175 being equal to $5 \times 5 \times 7$.

86. When a number will divide two or more numbers without a remainder, it is called their common measure.

Thus

Thus 5 is a common measure of 15 and 45, being contained in the former exactly 3 *times*, and in the latter exactly 9 *times*; again 4 is a common measure of 4 and 20, being contained in the former *once*, and in the latter 5 *times*.

85. Numbers, which admit of no common measure but unity, are said to be prime to one another.

Thus 4 and 9 are prime to each other.

86. From the above definitions it follows that no even number, except 2, is a prime number. For any other even number has the factor 2. Also that all prime numbers are prime to one another: For prime numbers have no factors; but a common measure to two numbers, that is greater than unity, must be a factor to one of them at least.

PROPOSITION.

87. If two, three, or more prime numbers be multiplied together, their product will not be divisible by any other prime number; or, in other words, the same number cannot be the product of two different sets of prime numbers.

Dem. If possible, let the number N be equal to $a \times b \times c \dots \times d$ and also to $p \times q \times r \dots \times s$, where the small letters denote prime numbers; a, b, c , &c. being all different from p, q, r , &c. and a being greater than p ; then since $abc\dots d = pqr\dots s$,

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$\frac{a}{qr..s} = \frac{p}{bc..d}$; let p be subtracted as often as possible from a , and the remainder be E ; and let $bc..d$

be subtracted *as often* from $qr..s$, and the remainder be R ; then * $\frac{a}{qr..s} = \frac{E}{R}$, and E is less than p by

unity at least: Let E be subtracted as often as possible from a , and R *as often* from $qr..s$, and the

remainders be F and S ; then $\frac{a}{qr..s} = \frac{F}{S}$, and F is

less than E by unity at least: Thus the subtractions may proceed till at last there will arise a

fraction equal to $\frac{a}{qr..s}$, whose numerator is unity;†

let this fraction be $\frac{1}{V}$, then since $\frac{a}{qr..s} = \frac{1}{V}$, $\frac{qr..s}{a}$

$= V$. It is proved therefore that if $pqr..s$ be divisible by a , so is $qr..s$: In the same manner it may be proved that if qrs be divisible by a , so is $r..s$; and thus we may proceed, whatever number of factors be supposed between r and s , till we

* If p be contained in a , n times and no oftener, $bc..d$ is contained in $qr..s$, n times and no oftener; let $a = np + E$ and $qr..s = nbc..d + R$; then since $a : p :: qr..s : bcd$,

$$a : np :: qr..s : nbcd$$

$$\text{and } a - np : a :: qr..s - nbcd : qr..s$$

$$\text{or } E : a :: R : qr..s$$

$$\therefore \frac{E}{R} = \frac{a}{qr..s}$$

† None of the remainders can be 0, till after the remainder 1 has arisen, because a , being a prime number, cannot be an exact multiple of E , F , &c. i. e. cannot be exhausted by subtracting from it as often as possible any number E or F , &c. greater than unity.

come to the last factor s , and find that s , a prime number, will be divisible by a , another prime number; which is absurd: $\therefore$ &c.

The factors a, b, c , &c. may some or all of them be equal; so also may p, q, r , &c.

If it be supposed that in case some of the factors were common to both sets, the products might be equal, let the two products be divided by all the common factors, and the products of the *remaining* factors would be equal; contrary to this demonstration; so that *if a number be the product of certain prime factors, p, q, r , &c. among which a is not found, it is not divisible by a .*

From this proposition flow immediately the following articles.

88. *The product of any prime numbers is prime to the product of any other prime numbers.*

Thus, if a, b, c, d , denote different prime numbers, ab is prime to cd ; a^2 to cd ; a^2 to c^2 ; $a^2 b^2$ to $c^2 d^2$, &c. &c.

89. *The product of any numbers, whether prime or composite, is prime to the product of any other numbers, each of which others is prime to each of the former.*

For the prime factors in the one product are all different from the prime factors in the other product.

Thus if each of the three A, B, C , be prime to each of the four D, E, F, G , $A B C$ is prime to $D E F G$.

90. *If two products, $A \times M, B \times N$, be equal, and A and B be prime to each other, M contains all the factors of B , and N all the factors of A .*

91. *If two numbers, a and b , be prime to each other, they are the least in their ratio.*

For if possible let $a : b : a - y : b - z$, then $a \times \overline{b - z} = b \times \overline{a - y}$, and since a contains none of the factors of b , $\overline{b - z}$ must contain them all; but $\overline{b - z}$ is less than b ; so that a less number is either exactly equal to, or else a multiple of a greater; which is absurd; $\therefore$ &c.

92. *If the numerator and denominator of a fraction be prime to each other, the fraction is in the lowest terms possible.**

For if a and b be prime to each other, and $\frac{a}{b}$

could be equal to $\frac{a - y}{b - z}$, we should have $a \times \overline{b - z}$

$= b \times \overline{a - y}$ contrary to the last art.

* The truth of this article was not assumed in art. 120, Vol. I. — nothing more was meant to be asserted there, than that the lowest terms, that could be had by expunging common factors from the numerator and denominator, were to be obtained by dividing by their greatest common measure.

93. *If two numbers a and b be prime to each other, any powers of them a^m , b^n , are prime to each other.*

For a^2 , a^3 , a^4 , &c. contain no other factors but a and the factors of a ; b^2 , b^3 , b^4 , &c. contain no other factors but b and the factors of b ; $\therefore$ if a and b have no common factor, neither have a^m and b^n , i. e. a^m is prime to b^n .

94. *Hence if two numbers be the least in their proportion, their squares, cubes, &c. will be the least in their's.*

95. *If a fraction, $\frac{a}{b}$, be in its lowest terms no other fraction can be equal to it, but such as has for its numerator some multiple of a , and for its denominator some multiple of b ; as $\frac{2a}{2b}$, $\frac{3a}{3b}$, &c.*

For if $\frac{a}{b} = \frac{p}{q}$, $aq = bp$, and since b contains none of the factors of a , p contains them all, and is therefore a multiple of a ; for the like reason q contains all the factors of b , and is therefore a multiple of b .

98. *If a whole number has not a whole number for its root, it has no root at all.*

For let p have no whole number for its square root,

root, and if possible let $\frac{a}{b}$ be its square root, or $\frac{a^2}{b^2} = p$, a and b being prime to each other; then $a^2 = pb^2$ $\therefore$ whatever factors are contained in b^2 , are contained in a^2 , contrary to the supposition, for a and b being prime to each other, a^2 and b^2 are prime to each other $\therefore$ &c.*

In general let q have no integral n th root, and if possible let $\frac{a}{b}$ be the n th root of q , or $\frac{a^n}{b^n} = q$,

a and b being prime to each other; then $a^n = qb^n$, and whatever factors are in b^n , are also in a^n , contrary to the supposition, for if a and b are prime to each other, so are a^n and b^n $\therefore$ &c.

* That no even number, which is not divisible by 4, as for example 2, or 6, or 10, can have a square root, may be demonstrated, independently of the foregoing articles, upon this principle,—that the square ($4n^2$) of any even number, $2n$, is divisible by 4.

For let p denote any odd number, and consequently $2p$ denote any even number not divisible by 4; and if possible let $\sqrt{2p} = \frac{a}{b}$, a & b being prime to each other; then $2p = \frac{a^2}{b^2}$ $\therefore 2pb^2 = a^2$ $\therefore a^2$ is an even number (because $2pb^2$ is;) $\therefore a$ is an even number (because an odd number multiplied into itself produces an odd number;) but a being even, a^2 is divisible by 4; $\therefore \frac{a^2}{2}$ or pb^2 , is even, $\therefore$ (since p is odd) b^2 is even $\therefore b$ is even; so that both a and b are even, $\therefore a$ and b are not prime to each other; being each divisible by 2, contrary to the supposition $\therefore$ &c. And particular demonstrations may upon similar principles be applied to other numbers and to higher roots.

Hence

Hence since the numbers 1, 2, 3, 4, &c. in their natural order, are the square roots of 1, 4, 9, 16, &c. no numbers but 1, 4, 9, 16, &c. have any square roots; there being no number whatsoever, whole or fractional, which multiplied into itself will produce any one of the intermediate numbers, 2, 3, 5, 7, 8, 10, &c.

Again, since 1, 2, 3, 4, &c. are the cube roots of 1, 8, 27, 64, &c. no numbers but 1, 8, 27, 64, &c. have any cube roots: And so on in the higher powers.

99. A fraction has no square root, unless its lowest terms be both square numbers, — no cube root, unless its lowest numbers be both cube numbers, — and so on.

For let $\frac{a}{b}$ be in its lowest terms, and let its m^{th} root be $\frac{c}{d}$, c and d being prime to each other; then $\frac{c^m}{d^m} = \frac{a}{b}$, and both these fractions are in their lowest terms $\therefore a$ is c^m and b is d^m or a and b are both of them m powers.

100. If two numbers, a and b be the least in their proportion, and there exists a mean proportional between them, they are both square numbers.

For let m denote the mean proportional between a and b ; then since $a : m :: m : b$, $a : b :: a^2 : m^2$; let e and f be the two least whole numbers in the ratio of a to m then $a : b :: e^2 : f^2$

f^2 , and both ratios are in their lowest terms; $\therefore$ a is e^2 , and b is f^2 , or a and b are both squares.

In a similar manner it may be shewn that if two numbers, a and b , be the least in their proportion, and there exist two mean proportionals between them, a and b are both cube numbers: And in general

101. *If two numbers, a and b , be the least in their proportion, and there exist $\overline{n-1}$ mean proportionals between them, a and b are n powers.*

For if $a : p :: p : q :: q : r :: \&c. \dots :: h : b$, where there are supposed to be n ratios, and consequently $\overline{n-1}$ mean proportionals, $p, q, \&c.$, a will be to b , as a^n to p^n ; let e and f be the two least whole numbers in the ratios of a to p ; then $a : b :: e^n : f^n$, both which ratios are in their lowest terms; $\therefore$ a is e^n , and b is f^n , or a and b are both n powers.

102. If a ratio in its lowest terms admits of no mean proportional number, neither does it in any other terms.

For let a and b be the lowest terms of a ratio, and pa, pb , denote any other terms; then if $\frac{m}{n}$ be a mean proportional between pa and pb , or $pa : \frac{m}{n} :: \frac{m}{n} : pb$, dividing each term by p , we shall

shall have $a : \frac{m}{np} :: \frac{m}{np} : b$, and there will be

a mean proportional, $\frac{m}{np}$, between a and b .

103. Hence we may construct as many incommensurable lines as we please: For we have only to take two lines in such a ratio, that the least numbers in the same ratio are not squares, and the mean proportional line between them will be incommensurable with either of them.

For let A and B denote any two lines, and M a mean proportional between them; then if A be commensurable with M , and consequently M with B , let A be to M as a to $\frac{m}{n}$, and M to B as $\frac{m}{n}$ to b ; then (art. 78) $a : b :: A : B$, and $a : \frac{m}{n} :: \frac{m}{n} : b$;

i. e. if M be commensurable with A , there is a mean proportional, $\frac{m}{n}$, between a and b , two numbers in the ratio of A to B : Consequently if A and B be taken in the ratio of two numbers that have not a mean proportional between them, M will be incommensurable with A or with B .*

* If M be incommensurable with A , it cannot be commensurable with B : For if M were certain parts of B , A would, by the definition of proportionality, be the same parts of M .

104. The Side (S) and diagonal (D) of a square are incommensurable with each other.

For let T denote a third proportional to S and D ; then, since by the 6th Book of Euclid, "if three straight lines be proportionals, as the first is to the third, so is any rectilinear figure upon the first, to a similar and similarly described rectilinear figure upon the second," it follows that S is to T , as the square upon S to the square upon D ; but these two squares are as 1 to 2, by the 47th of the 1st Book, and there is no mean proportional between 1 and 2; $\therefore S$ and T being in the ratio of two numbers, which have not a mean proportional between them, the mean proportional line, D , is incommensurable with S .

105. Any fraction, whose square exceeds 2, is too great for the value of the diameter in terms of the side; and any fraction, whose square falls short of 2, is too small for it.

For if $\frac{m^2}{n^2}$ be greater than 2, D cannot be so great

as $\frac{m}{n} S$, since then T would be as great as $\frac{m}{n} D$,

or $\frac{m}{n} \times \frac{m}{n} S$, or $\frac{m^2}{n^2} S$; i. e. greater than $2S$,

which it is not: And if $\frac{m^2}{n^2}$ be less than 2, D can-

not be so small as $\frac{m}{n} S$, since then T would be as

small

small as $\frac{m}{n^2} S$; i. e. smaller than $2 S$, which it is not. This limitation of the value of D in terms of S is denoted by the equation $D = \sqrt{2} \times S$, or by the proportion $S : D :: 1 : \sqrt{2}$.

Hence the decimal obtained in the operation of extracting the square root of 2, is in each figure the nearest possible defective value of D . Now that decimal is 1.414 &c. (see art. 205 Vol. I)

$\therefore D$ is greater than $\frac{14}{10} S$, but less than $\frac{15}{10} S$;

— greater than $\frac{141}{100} S$, but less than $\frac{142}{100} S$;

&c. agreeably to what was asserted in art. 74.

106. In the Chapter on Evolution it was shewn that a whole number, having no integral root, has no terminating decimal root: But it was not to be inferred from thence, that no fraction whatsoever could express the roots of such numbers, — since *there are* fractions which cannot be expressed by any terminate decimal. And if the decimal obtained in extracting the root of a number could after any figure become a circulating decimal, the value of the previous figures together with the value of the fraction, from which such circulating decimal might arise, would be the exact root of that number. i. e. for example, if the figures obtained in extracting the square root of a *certain number* were 3 . 4121212 &c. *ad*

infinitum, which is equal to $3.4 + .0121212$ &c. *ad infinitum*, the square root of that number would be $3.4 + \frac{2}{165}$, ($\frac{2}{165}$ being the fraction, from whence .01212 &c. arises.)

For if possible let N be a number, the figures of whose m^{th} root circulate; and let $\frac{S}{R}$ be equal to the sum of the value of the previous figures, and the value of the circulating part (the latter value being found by art. 66.) I say that in that case $\sqrt[m]{N} = \frac{S}{R}$, or $N = \frac{S^m}{R^m}$. For if N be not

equal to $\frac{S^m}{R^m}$, let $N = \frac{S^m}{R^m} \pm d = \frac{S^m \pm dR^m}{R^m}$;

then $\sqrt[m]{N} = \frac{\sqrt[m]{S^m \pm dR^m}}{R}$; i. e. the figures obtained for $\sqrt[m]{N}$ agree throughout with the decimal fraction obtained by dividing the figures of $\sqrt[m]{S^m \pm dR^m}$ by R ; but the figures of $\sqrt[m]{N}$ by hypothesis agree with those obtained by dividing S by R , $\therefore$ the figures of $\sqrt[m]{S^m \pm dR^m}$ are the figures of S , or $\sqrt[m]{S^m \pm dR^m} = \sqrt[m]{S^m}$, which is absurd.

$\therefore$ in that case $\sqrt[m]{N}$ would be equal to $\frac{S}{R}$; but because by hypothesis, N has no integral root, neither has it any fractional root, $\frac{S}{R}$; $\therefore$ The de-

cimal,

cimal, obtained by the operation of extracting the root of a whole number, cannot circulate.

105. When the radical sign is prefixt to a number, which has no root of the proposed denomination, the expression is called a *furd*, and sometimes an *irrational number*. Thus $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[4]{2}$, &c. $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$, &c. $\sqrt[3]{4}$, $\sqrt[4]{4}$, &c. &c. are called *furds*, or *irrational numbers*.

If it be asked what such expressions really mean, seeing that they have no numerical value, the answer is, that they denote mean proportionals between certain magnitudes, and arise in the investigation of the numerical value of those mean proportionals in terms of the extremes; that they prove them to be incommensurable with the extremes, and at the same time point out the limits of their value.: Thus, if between two magnitudes, A and B, we find the equation $B = \sqrt{2} \times A$, it shews that B is incommensurable with A, — at the same time it shews that if the square of a fraction, $\frac{m}{n}$ be greater than 2, B is not so great as $\frac{m}{n} A$, that if the square of a fraction, $\frac{p}{q}$, be less than 2, B is greater than $\frac{p}{q} A$.

But in order clearly to understand the nature of such expressions, and the legitimacy of the operations performed with them, and especially when
roots

roots are concerned of higher denominations than the square, it is necessary to consider the method of applying the algebraic notation and operations to the investigation of geometrical magnitudes.

SCHOLIUM.

106. VARIOUS objections have been urged by different writers against Euclid's definition of proportionality. Some have censured it as obscure and difficult to be understood; but that fault if it exists at all, is in the subtil nature of incommensurable magnitudes and not in Euclid's definition. Others have complained that they cannot, without a demonstration, discern whether quantities, that agree with the definition, are proportional or not; as if proportionality was something independent of definition. Others again, defining numbers to be proportional, when the quotient of the first divided by the second is equal to the quotient of the third divided by the fourth, have endeavoured to obviate the necessity for Euclid's, by extending this their definition to all magnitudes whatsoever; thereby endeavouring to confound the distinct natures of commensurable and incommensurable quantities; for what is the import of the expression, $\frac{A}{B}$, when A and B are incommensurable with each other?

A logical answer to this question would distinguish again what they thus endeavour to confound and blend together, and would satisfy the enquirer, that if he would have an accurate knowledge of the real nature of magnitudes, he must take the trouble of examining into the doctrine of Incommensurables, and at the same time shew him, that in point of conciseness nothing is gained by thus endeavouring to elude Euclid's distinctions, and in point of perspicuity much lost; the enquiry being of that nicety and subtilty, which cannot be easily dispatched in few words.

These and other objections are fully stated and answered in Dr. Barrow's Excellent Mathematical Lectures, (Lec. 22 and 23) to which the reader is referred if he be not satisfied with the above observations.

These disputes arise in a great measure from this circumstance: Logicians, not conversant with the really existing properties of magnitudes, enter upon the subject with preconceived * popular notions

* Incommensurability is a property of magnitudes totally unknown to the generality of those, who are unconversant with Mathematical demonstration. If such be asked whether the side and diagonal of one square be in the same proportion as the side and diagonal of another, they will answer in the affirmative: But if more closely examined as to what they mean by proportionality, their explanation will shew they are not aware that the side and diagonal of a square have no common measure whatsoever. For, as Aristotle observes,

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tions of proportionality, and finding at the very threshold a definition, whose use they know not, and which seems to them to render obscure, what they consider as a well known property, they conclude it to be at least an unnecessary bar to the attainment of science, and vainly employ their logic to overturn its authority. Whereas, if they would but first examine into the properties of magnitudes, they would be led of themselves to discern the utility of such a definition, and to see how totally from the purpose (not to say unmeaning) are all their objections.

The founders of Geometrical science certainly did not previously lay down this definition of proportion, (devising it from abstract considerations) and from thence investigate the properties of magnitudes; but first discovering their properties by some more circuitous rout, were led, by a certain analogy and connexion observed in them, to see how a tautologous and perplexing repetition of words might be avoided, by giving a general name to a certain property; for that is the only use of the words, "ratio," "equal ratio," "duplicate" ratio, &c. &c. But when the truths discovered are

"it seems strange to the many that the greater of two things of the same kind cannot be exhausted by some *very small* measure of the less." Plato severely reproaches the mass of his Countrymen for their erroneous notions on this subject, lamenting their ignorance of so common a property, as shameful, and more fit for brute beasts than rational men.

reduced

reduced to a science, the general names and definitions appear first, and the student is expected not to dispute their utility, till he be acquainted with the properties whose investigation they are designed to expedite. Yet as it is found disagreeable to the mind to acquiesce in any thing, of whose utility it is totally ignorant, it would be better perhaps, if those, who undertake to expound science, would relax somewhat from the rules of strict method, and indulge their disciples with some account of the use of their definitions.

This object has not been neglected in the foregoing account of incommensurables. And it is hoped that the method pursued in it will be found of itself in a great measure to obviate such objections, as those above stated. If any trifling faults with respect to arrangement be observed, the following consideration is submitted as an apology:—That the author has been able to meet with nothing on this subject of incommensurables, excepting Saunderson's treatise in his *Algebra*, at once full and logical. Saunderson's method he did at first intend to adopt; but the want of one general proposition in it, with an independent demonstration, from which the whole doctrine would immediately flow in such a manner, that whoever had that proposition fully established in his mind, could not fail of being able to demonstrate the existence of incommensurables, induced him to set it aside, and seek for an inde-

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pendent

pendent demonstration of the property of numbers, that *the product of any prime numbers is not divisible by any other prime number.*

It may appear rash to dissent from what has been so long and so generally received, but truth is more sacred than Euclid; though if any authority could be allowed in Mathematics, it would be the authority of him, the unrivalled Father of that science: But it must be confessed that in Euclid's doctrine of numbers in his 7th Book there is a broken link, which destroys the chain of reasoning. This link is the 21st proposition, which upon examination the reader will find to be undemonstrated.

The proposition asserts, and truly asserts, that if two numbers a and b be the least in their proportion, they will *equally* measure any other two numbers c and d , that are in the same proportion; the less being an aliquot part of the less, and the greater the *same* aliquot part of the greater. As Maclaurin in his chapter on surds has followed the same method of demonstration, which Euclid employs, I shall present it to the reader in his words.

“For if a and b are not aliquot parts of c and d , they must contain the same number of the same kind of parts of c and d , and therefore dividing a into parts of c , and b into an equal number of like parts of d , and calling one of the first m , and one of

of the latter n ; then as m is to n so will the sum of all the m s be to the sum of all the n s; that is $m : n :: a : b$; therefore a and b will not be the least in the same proportion; against the supposition. Therefore a and b must be aliquot parts of c and d ."

Now this demonstration takes for granted what more requires a proof than the proposition itself. For though it follows from the nature of proportional numbers that a must be the same parts of c that b is of d , a being $\frac{a}{c}$ *ths* of c and b being $\frac{a}{c}$ *ths* of d , yet it does not appear that b can be divided into simple *cth* parts of d . For example 7 is $\frac{5}{15}$ *ths* of 21; yet 7 cannot be divided into 5 integral parts, each a 15th part of 21.—Neither supposing $\frac{a}{c}$ reduced to any other terms, $\frac{p}{q}$, do we know that because a is equal to $\frac{p}{q} c$, and b equal to $\frac{p}{q} d$, a and b can, one or both, be divided into p integral parts. Therefore the demonstration fails. But it was not on this account that Euclid's propositions were not adopted here, for the deficiency of this demonstration * might easily be supplied;

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* As for example by the following demonstration, where nothing is assumed but what is warranted by the previous propositions of the 7th Book.

plied; but because his method and order, however excellent in themselves, are not adapted to the present state of Algebraic science.

If c be not a multiple of a , and consequently d not a multiple of b , let the greatest multiple of a , that is contained in c , be na , and the remainder of c be r ; then the greatest multiple of b , that is contained in d , is nb ; let the remainder of d be s ; then r and s are less than a and b respectively;

and since $a : b :: c : d :: na + r : nb + s$

and $a : b :: na : nb$, $na : nb :: na + r : nb + s$

$\therefore na : nb :: r : s \therefore a : b :: r : s$, and a and b are not the least in their proportion, contrary to the supposition
 $\therefore$ &c.

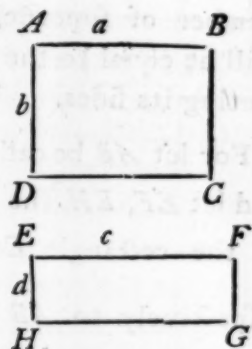
CHAPTER IV.

OF THE APPLICATION OF ALGEBRA TO RECTILINEAR GEOMETRY.

107. IF there be two rectangles $ABCD$, $EFGH$, and any four numbers, a, b, c, d , such that

$$AB : EF :: a : c$$

and $AD : EH :: b : d$,



the rectangle AC will be to the rectangle EG in the same ratio as ab to cd .

For, by art. 78, the ratio compounded of the two ratios $AB : EF$ and $AD : EH$ is equal to the ratio compounded of the two ratios $a : c$ and $b : d$; the former compound ratio is equal to the ratio $AC : EG$ by Euc. B. 6 p. 23, and the latter is equal to the ratio $ab : cd$ (see ratios art. 275) $\therefore AC : EG :: ab : cd$.

Note.

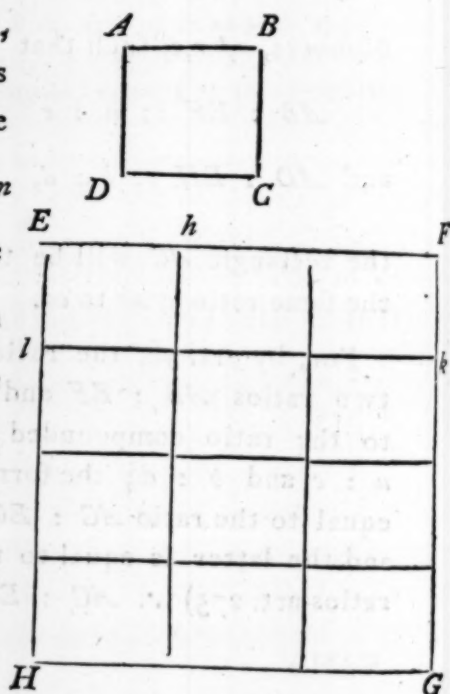
Note. It is not necessary for a and b to be in the ratio of the sides AB, AC ; nor c and d in that of the sides EF, EG .

The proposition may be briefly expressed thus:
The magnitude of a rectangle varies as the product of the two numbers denoting the length of its sides.

108. Cor. If any line at pleasure be called 1, and other lines proportionably, and that surface, which is equal to the square on the line 1 be also called 1, and other surfaces proportionably, the number of superficial unities in any rectangle will be equal to the product of the numbers representing its sides.

For let AB be called 1, and let EF, EH , the sides of the rectangle EG be respectively to AB as m and n to 1; then, by the last art., the square on $AB : EG :: 1 \times 1 : mn :: 1 : mn$, or EG contains mn of the squares on AB .

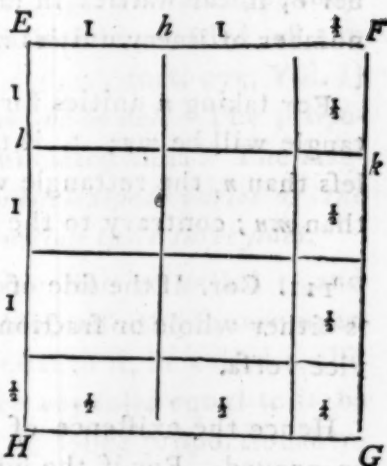
This Cor. may easily be demonstrated, without



referring to the doctrine of ratios, as follows. Divide the sides of EG into unities Eh , &c. Fk , &c. El , &c. and join the opposite points of division:

Then Eh contains m square unities; and EG contains Eh , n times, and

$\therefore$ contains n times m square unities, or mn square unities.



109. Conversely to art. 107, if the two rectangles be as ab to cd , and AB be to EF as a to c , AD is to EH as b to d .

For since by hypothesis the sum of the two ratios $AB : EF$ and $AD : EH$ is equal to the sum of the two ratios $a : c$ and $b : d$, subtracting the equal ratios from each sum, the remaining ratios are equal.

110. Conversely to art. 108, if the number of square unities in a rectangle be mn , and the number

ber of linear unities in one of the sides be m , the number of linear unities in the other side is n .

For taking n unities for the other side, the rectangle will be mn ; $\therefore$ if the side were greater or less than n , the rectangle would be greater or less than mn ; contrary to the supposition.

III. Cor. If the side of a square be m , where m is either whole or fractional, the square is m^2 , and vice versa.

Hence the existence of incommensurables may be proved. For if the side of an equilateral rectangle be commensurable with the linear unity, let it $= m$; then, m^2 , the number denoting how many square unities are contained in the rectangle, admits of a square root; $\therefore$ in case this number does not admit of a square root, the side is not commensurable with the linear unity.

112. If there be two rectangular Parallelepipeds $ABCD$, $EFGH$, (see plate, fig. 1) and any six a, b, c, d, e, f , such that

$$AB : EF :: a : d,$$

$$BC : FG :: b : e,$$

$$\text{and } CD : GH :: c : f,$$

the solid AD will be to the solid EH in the same ratio as abc to def .

For the solids are to each other in the ratio compounded of the ratios of their three sides

(Simson's

(Simson's Euc. B. XI, pr. D;) $\therefore$ by art. 79, they are to each other in a ratio compounded of the ratios $a : d$, $b : e$ and $c : f$; $\therefore$ (art. 275, Vol. I) they are to each other as abc to def . The proposition may be briefly enunciated thus: The magnitude of *rectangular parallelepipeds* varies as the product of the numbers denoting their three sides.

113. Cor. Hence if any line be called 1, and other lines proportionably, also the square upon the line 1, or any area equal to it, be called 1, also the cube on the line 1, or any solid equal to it, be called 1, and other areas & solids proportionably, the number of solid unities in a rectangular parallelepiped will be equal to the product of the numbers denoting the three sides.

For if $a = b = c = 1$, AD will be the cubical unity, and we shall have $AD : EH :: 1 : def$, or EH will contain AD def times.

This corollary may also be demonstrated in a similar manner as art. 108. thus:

The plane EG contains de square unities, $\therefore$ if GH be divided into unities, and the whole solid be divided into lamina by a plane passing parallel to EG through each point of division, the first of these lamina will contain de cubical unities, and since there will be f of such lamina, the whole solid contains f times de cubical unities.

114. Conversely, if two rectangular parallelepipeds be to each other as abc to def , and a side in one be to a side in the other as a to d , the rectangles under the remaining sides will be to each

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other

other as bc to ef : or if the rectangles under two sides in each be to each other as ab to de , the remaining sides will be to each other as c to f .

115. If four magnitudes A, B, C, D be proportional, and C be to D as the *number* c to the *number* d , $A = \frac{c}{d} \times B$.

For A is the same parts of B , that C is of D ; but $C = \frac{c}{d} \cdot D$, $\therefore A = \frac{c}{d} \times B$.

116. If each line be denoted by some letter, — the rectangle under any two lines, by the junction of the two letters productwise, — the solid parallelepiped contained under any three lines, by the junction of the three letters productwise, — any term in a proportion, by the same expression, as would, if the letters denoted numbers, be the numerical value of that term; e. g. a mean proportional between a and b , by $\sqrt{ab}$; a fourth proportional to a , b , and c , by $\frac{bc}{a}$; the first of three mean proportionals between a and b , by $\sqrt[3]{a^2b}$; &c. &c. — any ratio which is compounded of other ratios, the same as in algebraic operations, viz. by the ratio between the product of the antecedents and the product of the consequents; and therefore, agreeably to the fourth

head,

head, any magnitude, that is to another in a ratio compounded of certain ratios, by that other, multiplied into the product of the antecedents of those ratios, and divided by the product of their consequents; so that, for example, if a be to b in a ratio compounded of the ratios $c : d$, $e : f$, $g : h$, and $k : l$, it will be denoted by the proportion $a : b :: cegk : dfhl$, and a will be denoted by $b \times \frac{cegk}{dfhl}$ or $\frac{bcegk}{dfhl}$, — the subtrac-

tion of any ratio from another, (as in algebra) by dividing the antecedent of that other by the antecedent of the ratio to be subtracted, and its consequent by the consequent; e. g. denoting the sum of the ratios $abc : def$ — the sum of the ratios $gh : kl$, by $\frac{abc}{gh} : \frac{def}{kl}$ — and lastly the half

of any ratio by the ratio of the square roots of its terms; the third part of any ratio by the ratio of the cube roots of its terms, and so on; e. g. denoting half the ratio $abcd : efgh$ by $\sqrt{abcd} : \sqrt{efgh}$; an n^{th} part of the ratio $abc : def$ by $\sqrt[n]{abc} :$

$\sqrt[n]{def}$; a 4th of the ratio $a^3b : c^4$ by $\sqrt[4]{a^3b} :$

$\sqrt[4]{c^4}$, or $\sqrt[4]{a^3b} : c$; &c. &c. then whatever algebraic operations are performed with equations or proportions derived from geometrical questions, the resulting equations and proportions will be true, in whatever light, agreeably to the above hypothesis, the terms be considered; and if any

equation arises, in which the terms denote neither lines surfaces nor solids, such equation will be true, if considered as denoting either that the sum of the ratios of the factors on one side to the factors on the other side, is a ratio of equality, or that the sum of the ratios of the factors on one side to any proposed magnitude, is equal to the sum of the ratios of the factors on the other side to the same magnitude; e. g. if we find the equation $abcd = efgh$, it will prove that the sum of the ratios $abcd : efgh$ is a ratio of equality; or it will prove that if we take any line M , the sum of the ratios $a : M, b : M, c : M$ and $d : M$ is equal to the sum of the ratios $e : M, f : M, g : M$ and $h : M$.

In order to * prove the truth of these assertions generally, it will be necessary to consider some of those

* To prove it in any particular case is very easy; and if all magnitudes were commensurable, it *might* be generally proved by substituting for magnitudes proportional numbers; for then the results, being true when considered as numbers, would be true if considered as geometrical magnitudes, on account of the analogy between products and rectangles or parallelepipeds; e. g. if $ab = \frac{cdef}{gh}$, then, since as numbers, $ab : cd :: ef : gh$, and $ab : cd :: \text{rectangle } ab : \text{rectangle } cd$, $\text{rectangle } ab : \text{rectangle } cd :: \text{number } ef : \text{number } gh$; and $\text{rectangle } ab = \text{rectangle } cd \times \text{the number } \frac{ef}{gh}$: Yet even then it would be highly useful to consider the operations at first without any reference to arithmetical computation.

But

those properties of ratios, which are analogous to the properties of numbers.

Note, in the following articles the ratio between any two terms, that are not simple letters, denotes a compound ratio, as explained above. And the signs $+$ & $-$ between two ratios denotes the addition and subtraction of the ratios, not of their terms.

117. *Any ratio, compounded of ratios of equality, is itself a ratio of equality.**

For

But the necessity for surds in applying algebraical notation to Geometry destroys the possibility of considering the quantities as numbers; and the operations with surds could not be easily explained without referring to geometry, nor if they could, would the legitimacy of applying these operations to geometrical enquiries less require to be proved generally: Thus, supposing the four numbers, a, b, c, d proportional, it will be found very troublesome to explain, independently of geometry, what is meant by the proportion $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}$ or by the equation $\sqrt{a} \times \sqrt{d} = \sqrt{b} \times \sqrt{c}$, or by the equation $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, provided these roots be irreducible surds; and when that is explained, it will remain to be proved, that two lines, whose squares are as a to b , are in the same proportion as two lines, whose squares are as c to d ; and this is only a particular and easy case of what must be explained generally.

* The necessity and force of demonstrations of this kind concerning ratios are often but obscurely seen by learners, on account of their not sufficiently attending to the exact import of words. The ratio, which one magnitude has to another, is said to be compounded of any ratios that can be formed by making the former of the two magnitudes the first antecedent, and the latter the last consequent, taking for the other consequents any magnitudes whatsoever, (of the same kind) but making each consequent the antecedent of the next ratio. Thus, if A, B, C, D , &c. denote any magnitudes of the same kind

For let there be any ratios of equality $a : a$, $b : b$, $c : c$, &c. then, since these ratios are respectively equal to the ratios $a : a$, $a : a$, &c. $a : a$, &c. the ratio compounded of them (by which is meant the ratio between the first antecedent and the last consequent of these other ratios,) is the ratio $a : a$.

118. *If to or from any ratio, $A : B$, there be added or subtracted a ratio of equality, the resulting ratio will be equal to $A : B$; or a ratio is not changed by the addition or subtraction of a ratio of equality.*

For let $a : a$ be the ratio of equality, and let $A : B = a : r$; then since

$$A : B :: a : r$$

and $a : a :: r : r$, the ratio compounded of the ratios $A : B$, and $a : a$ is $a : r$; the same as the ratio $A : B$.

Hence $A : B - a : a = A : B$, because adding to both sides the ratio $a : a$, we have $A : B = A : B - a : a = A : B$.

kind, A is said to have to B the ratio compounded of, or the sum of the ratios A to C and C to B ; or the ratios A to C , C to D and D to B ; or the ratios A to C , C to D , D to E and E to B ; or &c. &c.: And by a ratio compounded of ratios, whose antecedents and consequents do not follow that law, is meant the ratio between two magnitudes, which is, according to that law, compounded of certain ratios, each respectively equal to each of the former.

Thus, if $a : b : m : n$, and $c : d :: n : q$, by the ratio compounded of the ratios $a : b$, $c : d$, is meant the ratio $m : q$; and it is denoted by $ac : bd$.

119. *If*

119. *If the antecedents or the consequents of any simple ratios have their places changed one among another, the sum of the ratios remains the same.*

For 1st let there be two simple ratios $a : b$ and $c : d$; and let $c : d = b : r$; then
 $c : d = d : r$ (Euc. B. 5: pr. 16)
 $\therefore a : b + c : d = a : b + b : r = a : r$,
 and $a : d + c : b = a : d + d : r = a : r$,
 $\therefore a : b + c : d = a : d + c : b$; or any two consequents may mutually change places.

2. Let there be any number of ratios $a : b, c : d, e : f, \&c$; then since $a : b + c : d = a : d + c : b$, the sum of the ratios is equal to $a : d + c : b + e : f + \&c$. Again since $c : b + e : f = c : f + e : b$, the sum of the ratios is equal to $a : d + c : f + e : b + \&c$. where the consequents are changed not mutually; $\therefore$ any one of the antecedents may take for its consequent any one of the consequents.

120. *Hence the order of the factors in any expression may be changed ad libitum.* Thus $abc : def = abc : fde = bca : fed = \&c$.

121. *If to any ratio its inverse be added, the sum is a ratio of equality.*

For $abc : def + def : abc = abcdef : d : fabc =$ (by last art.) $abcdef : abcdef =$ (art. 117) a ratio of equality.

122. *The*

122. *The subtraction of any ratio from another produces the same ratio as the addition of the inverse ratio.*

$$\begin{aligned} \text{For } A : B - c : d &= A : B - c : d \\ - d : c + d : c &= A : B - c : c - d : d + \\ d : c \text{ (art. 119)} &= A : B + d : c \text{ (art. 118)} \end{aligned}$$

123. *The sum of the ratios of the magnitudes in the extremes of a proportion to the magnitudes in the means is a ratio of equality.*

Let $abc \&c.$ be to $def \&c.$ as $ghk, \&c. lmn \&c.$ I say that $abclmn \&c : defghk \&c.$ is a ratio of equality.

For to the ratio $abc : def$, and also to its equal, the ratio $lmn : ghk$, adding the ratio $lmn : ghk$, we shall have $abclmn : defghk :: ghklmn : lmnghk :: ghklmn : ghklmn$ (art. 120) but (by art. 117) $ghklmn : ghklmn$ is a ratio of equality, $\therefore abclmn : defghk$ is a ratio of equality.

124. *Conversely, if a compound ratio be a ratio of equality, then the ratio of any one of the antecedents, to any one of the consequents, is equal to the sum of the ratios of the remaining consequents to the remaining antecedents: Also, the sum of the ratios of any two or more of the antecedents, to any two or more of the consequents, is equal to the sum of the ratios of the remaining consequents, to the remaining antecedents.*

For

For let $abclmn : defghk$ be a ratio of equality; let a be to d as $efghk$ to $sclmn$; then, by the last, the sum of the ratios $sc.lmna : efghkd$ is a ratio of equality; so also is the sum of the ratios $bcdlmna : efghkd$; $\therefore$ subtracting the common ratios from each sum, $s : e = b : e$, $\therefore s = b$, and $a : d :: efghk : bclmn$. And in the same manner it may be proved that $ab : de :: fghk : clmn$; that $abl : fgl :: deh : cmn$; &c. Moreover, it is evident that the demonstration does not depend on the number of factors.

125. Cor. 1. *Any proportion is true, if the terms be taken alternately.*

For let $abc : def :: ghk : lmn$, then $abclmn : defghk$ is a ratio of equality, $\therefore abc : ghk :: def : lmn$.

Note. The terms must all have the same number of factors, otherwise the alternate proportion would have no meaning.

126. Cor. 2. *Any factor in one extreme may change places with any factor in the other extreme; and the same in the means.*

For if $abc : def :: ghk : lmn$, $abclmn = defghk$, $\therefore abl : def :: ghk : mnt$, or c and l may change places.

127. Cor. 3. *The sum of the ratios, which each magnitude in the extremes of a proportion has to any proposed magnitude of the same kind, is equal to the*

O

sum

sum of the ratios, which each magnitude in the means has to that same magnitude.

For if $abc : def :: ghk : lmn$, $abclmn : defghk$ is a ratio of equality; $\therefore$ taking M any magnitude whatsoever, $abclmn : defghk :: MMMMMM : MMMMMM$ and alternately, $abclmn : M^6 :: defghk : M^6$.

This property is denoted by the equation $abclmn = defghk$. (see art. 116.)

128. Cor. 4. *Conversely, an equation of this kind may be thrown into a proportion.*

Thus if $abc = def$, i. e. if $abc : M^3 = def : M^3$, then (art. 125) $abc : def :: M^3 : M^3$, $\therefore abc : def$ is a ratio of equality, $\therefore$ (art. 124) $a : d :: bc : ef$, or $ab : fd :: e : c$ or &c.

129. *The ratio between two fractions, each of which have a factor common to the numerator and denominator, is not altered by expunging the two common factors.*

Thus the ratio $\frac{ab}{b} : \frac{cd}{c}$, being equal to $a : d$

$+ b : c - b : c$, is equal to $a : d$.

130. *If two factors in a ratio, one being in the antecedent and the other in the consequent, be multiplied or divided by the same number, the ratio will not be altered.*

For

For the ratio between any two factors and between their equi-multiples or equi-sub-multiples is the same,

131. If one factor in any term be multiplied by any number, and another factor be divided by the same number, the ratio is not altered.

For the ratio $ab : cd$ is equal to the ratio $na \times b : nc \times d$, $= na \times b : d \times nc = na \times \frac{b}{n} : d \times$

$$\frac{nc}{n} = na \times \frac{b}{n} : cd.$$

132. Hence, any numerical co-efficient may be transferred from one factor to another.

For $na \times b : cd = a \times nb : cd$.

Also, the numerical co-efficients may be collected together, and their product be prefixt as a co-efficient to the first letter :

Thus $2a \times 3b : cd = 2 \times 3 a \times b : cd = 6ab : cd$.

133. The numerator and denominator of a fraction may be multiplied or divided by any, the same number.

For let n denote any number, whole or fractional, then $\frac{a}{b} : \frac{c}{d} = a : c - b : d = na : nc + nd : nb$ (art. 122) $= na : nb + nd : nc = na :$

$$nb+d : c = (\text{art. 119}) na : c+d : nb = na : c$$

$$- nb : d = \frac{na}{nb} : \frac{c}{d}, \text{ by notation.}$$

$$\text{Aliter. } \frac{a}{b} : \frac{c}{d} = \frac{abd}{b} : \frac{cbd}{d} = ad : cb =$$

$$nad : nbc = \frac{nad}{nd} : \frac{nbc}{nd} = \frac{na}{nb} : \frac{c}{d}.$$

134. In a fractional expression, any numerical co-efficient in the numerator is equivalent to its reciprocal in the denominator; and vice versa.

For expunging a co-efficient from the numerator, and multiplying its reciprocal into the denominator, is dividing both numerator and denominator by that co-efficient.

$$\text{Thus } \frac{nab}{c} : \frac{de}{f} = \frac{ab}{\frac{1}{n}c} : \frac{de}{f}.$$

135. In multiplying or dividing one expression by another, the algebraic rules may be employed.

$$\text{Thus } \frac{ab}{c} \times \frac{d}{e} : \frac{f^2}{g^2} = \frac{abd}{ce} : \frac{f^2}{g^2}; \sqrt{ab} \times \sqrt{cd} : ae \text{ (which is by definition half the sum of the ratios } ab : a^2 \text{ \& } cd : e^2, \text{)} \text{ is equal to half } abcd : a^2e^2 = (\text{by def.}) \sqrt{abcd} : ae; \frac{ab}{c} \div \frac{d}{e} : \frac{f^2}{g^2} \div$$

$$\frac{h}{l} \text{ or } \frac{ab}{c} : \frac{f^2}{g^2} = \frac{d}{e} : \frac{h}{l} = \frac{ab}{c} : \frac{f^2}{g^2} + \frac{e}{d} : \frac{l}{h} =$$

(by

(by notation) $\frac{ab}{c} \times \frac{e}{d} : \frac{f^2}{g} \times \frac{1}{h}$.

136. From the above articles it follows, that all the operations of multiplication, division, involution and evolution, considered as the composition and decomposition of ratios, may be applied to geometrical magnitudes.

For multiplication is then the addition of ratios, equals to equals; division is the subtraction of ratios, equals from equals; involution is the addition of equal ratios together, or in other words, the multiplication of ratios; and evolution is the reverse, or the division of ratios.

137. The results of any operation will be true, if the terms be considered as geometrical magnitudes, according to art. 116.

To prove this, it is only necessary to shew that the ratio between any two expressions, of such a form as to denote geometrical magnitudes, is, when considered as a compound ratio, the same as the ratio between the terms which they denote in proportions; e. g. to prove that $\frac{ab}{c} : \frac{de}{f}$ considered as the sum of the ratios $a : d$ and $b : e$ diminished by the ratio $c : f$ gives the ratio, which

which a 4th proportional to c , a and b has to a 4th proportional to f , d and e .

Let then A and B denote any two geometrical magnitudes of the same kind, and $C : D, E : F$ any two ratios simple or compound; then $A \times \frac{D}{C}$

and $B \times \frac{F}{E}$ will denote any two geometrical magnitudes, (not under a radical form,) that may be derived from proportions; let them denote R and S , i. e. let R and S be 4th proportionals to C, D , and A and to E, F , and B respectively; then since $C : D :: A : R$, and $E : F :: B : S$, $CR : DA$ and $ES : FB$ are ratios of equality (art. 123) $\therefore CR : DA :: ES : FB$, and (art. 125) $CR : ES :: DA : FB$, and $R : S :: \frac{DA}{C} : \frac{FB}{E}$, $\therefore$ if $\frac{DA}{C}$ be taken to denote R , $\frac{FB}{E}$ will denote S .

Again, let A, G, B, H denote any four magnitudes; then $\sqrt[m]{A^{m-1}G}$ and $\sqrt[m]{B^{m-1}H}$ will denote any two magnitudes, that are the first of $m-1$ mean proportionals between other two magnitudes; let them denote R and S ; then $A : R :: R : G$, and $B : S :: S : H$; and as compound ratios $A : G :: A^m : R^m$, and $B : H :: B^m : S^m$; hence, as before, $R^m : S^m :: A^{m-1}G : B^{m-1}H$, $\therefore$ the m^{th} parts of these ratios are

are equal, or $R : S :: \sqrt[m]{A^{m-1}G} : \sqrt[m]{B^{m-1}H}$,
 $\therefore$ if $\sqrt[m]{A^{m-1}G}$ be taken to denote R , $\sqrt[m]{B^{m-1}H}$
 will denote S .

If the two expressions be not similar, with respect to the number of factors in their numerators and denominators, and consequently denote no compound ratio, the demonstration may be extended to them by adding to the numerator and denominator of the less compound of the two, a competent number of equal factors to reduce them to similarity, which will not alter the value of that expression considered as a geometrical magnitude.

Thus if $a^2 : b^2 :: c : d$, and $e : f : g : h$,
 to shew that $\frac{cb^2}{a^2}$ and $\frac{gf}{e}$ may be employed for d

and h respectively, add to the numerator and denominator of the latter the factor e , or, which is the same thing, for the latter proportion substitute the proportion $e^2 : ef :: g : h$; then the ratio $\frac{cb^2}{a^2} : \frac{gfe}{e^2}$ will be the same ratio as $d : h$,

and any equation or proportion between geometrical magnitudes, derived from it, will be true; but any equation or proportion derived from $\frac{cb^2}{a^2} : \frac{gf}{e}$ may be reduced, without altering the value of the magnitudes concerned, to one, that might have been derived from $\frac{cb^2}{a^2} : \frac{gfe}{e^2}$; there-

fore

fore $\frac{cb^2}{a^2} : \frac{gf}{c}$ may be employed.

For example, as geometrical magnitudes let $\frac{cb^2}{a^2} : \frac{gf}{c} :: \sqrt{mn} : \sqrt{pq}$; then squaring each term, and reducing, we shall find this value for c^2 viz. $c^2 = \frac{mng^2f^2a^4}{pqe^2b^4}$, or c^2 a 4th proportional to pqe^2b^4 , $g^2f^2a^4$ and mn ; a true proportion, because if we employ $\frac{gfe}{c^2}$, we shall find c^2 a 4th proportional to pqe^2b^4 , $g^2f^2e^2a^4$ and mn ; which is the same as the former 4th proportional, because the ratios $pqe^2b^4 : g^2f^2a^4$ and $pqe^2b^4 : g^2f^2e^2a^4$ are equal.

Since then it appears from Euclid that the ratio between any two products, each consisting of two or three simple factors, is, when considered as a compound ratio, the same as the ratio between the rectangles, or rectangular parallelepipeds, contained under those factors; and it has been shewn that instead of any magnitude under a simple form, such as a , ab , alc , its algebraical value, derived from any proportion, may be substituted, it follows that the operations of multiplication, division, involution, and evolution may be applied to geometrical magnitudes, if the results be considered as explained in art. 116.

138. It remains to be proved, 1st that the operations of addition or subtraction may be employed in geometrical equations; 2^{dly}. that if any magnitude be the sum or difference of two or more magnitudes, that sum or difference may be employed in multiplication instead of the magnitude itself, agreeably to the algebraic rule "*like signs give + unlike —*"; and thirdly, that if the numerator of an expression consists of more than one term, the expression is equal to the sum of the quotients of each term of the numerator, divided by the denominator.

With respect to the first head, it is evident that the rules for addition and subtraction are applicable to any magnitudes whatsoever of the same kind, inasmuch as they depend on this principle, that if equals be added to or subtracted from equals, the sums or remainders are equal.

The second head may be proved by means of the 1st proposition of the 2^d book of Euclid and of three other propositions, which may be demonstrated in a similar manner. That proposition shews, that if $a = m + n$, the rectangle $ab = mb + nb$; in a similar* manner it may be shewn, that if $a = m - n$, $ab = mb - nb$, and hence, as in Algebra, it may be shewn, that if a and b , one or both, be the sums or differences of any number of

* Or thus without any construction: If $a = m - n$, $m = a + n$, and $mb = ab + nb$ (by the 1st prop.) $\therefore ab = mb - nb$.

parts whatsoever, the value of the rectangle ab is had by taking the rectangles under each part of a and every part of b , and prefixing $+$ to those rectangles whose two sides have like signs, and $-$ to those whose two sides have unlike signs.* Also, by demonstrations similar to the above, it may be shewn of rectangular parallelepipeds, that if $a = m \pm n$, $ade = mde \pm nde$, and hence that if the factors of a rectangular parallelepiped be composed of any number of parts, its value is had by multiplying the values of the factors together as in Algebra.

The third head is easily proved from the preceding, being its converse. For if $a = m + n$, and consequently $\frac{ad}{e} = \frac{md + nd}{e}$, $\frac{ad}{e}$ will be equal to

• It might seem at first sight that the 1st proposition of the 2d Book of Euclid would follow as a consequence of the analogy between rectangles and products, and that there is therefore no necessity for Euclid's demonstration by a construction; but that analogy could not easily be applied to any other case than where the two lines and the two parts, into which one of them is divided, are all commensurable with one another; and therefore would not serve generally: But this 1st proposition being demonstrated as in Euclid, and the general corollary concerning the rectangle under any compound lines being deduced from it, the nine following propositions (and others similar to them) may be demonstrated without any construction. Such demonstrations do not require the Algebraic notation, but might proceed in the same manner as Euclid's demonstration of the 12th prop. of the 2d Book; they are not to be considered as numerical, for the use of that notation is only to abridge the number of words.

$\frac{md}{e} + \frac{nd}{e}$; because, multiplying both sides by e , the rectangle under $\frac{ad}{e}$ and e , i. e. the rectangle ad , is equal to the rectangle under $\frac{md}{e}$ and $e +$ the rectangle under $\frac{nd}{e}$ and e , i. e. to the sum of the rectangles $md + nd$.

Moreover, nothing untrue can arise from extending this article to expressions, which do not denote geometrical magnitudes: Because each step in the process may, by multiplication or division, be reduced to a true equation between geometrical magnitudes, and the result at last be the same as before.

Thus, if $a = m + n$; we may employ the equation $adef = mdef + ndef$, and conclude that the equation $\frac{adef}{g} = \frac{mdef}{g} + \frac{ndef}{g}$, or any other equation derived from it, whole terms denote geometrical magnitudes, is true. For $ad = md + nd$, $\therefore ad \times \frac{ef}{g} = md \times \frac{ef}{g} + nd \times \frac{ef}{g}$
 or $\frac{adef}{g} = \frac{mdef}{g} + \frac{ndef}{g}$

139. It may be concluded therefore, that if lines, surfaces, solids, and compound ratios be denoted as explained in article 116, all the operations of Algebra may be applied to the equations or proportions, derived from the data or conditions of any geometrical question.

Example 1. Theorem. (the 9th proposition of the 2d Book of Euclid.)

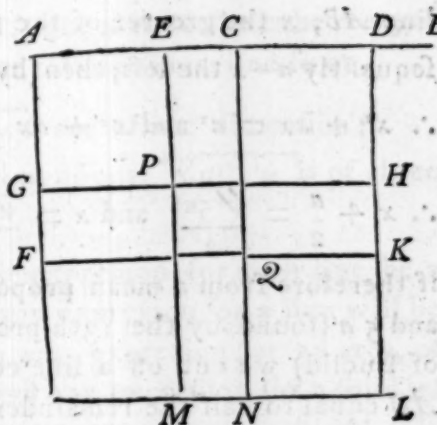
If a strait line AB be divided into two equal parts, (at the point C ,) and also into two unequal parts, (at the point D ,) the squares of the two unequal parts (square of AD + square of DB) are together double the square of half the line, and of the square of the line between the points of section. (double of the squares of AC and CD).

$\begin{array}{ccccccc} & A & & C & & D & & B \\ \hline & & & & & & & \end{array}$

Let a denote AC , and b denote CD ; then $AD = a+b$ and $DB = a-b$; $\therefore$ the square of AD + the square of $DB = \overline{a+b}^2 + \overline{a-b}^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2a^2 + 2b^2 =$ double the squares of AC and CD . QED.

This proposition may be demonstrated by a construction as follows:

On AD describe the square AL; take CE = DB; AF = AC; AG = AE = CD; through E and C draw EM, CN parallel to DL, and through G



and F draw GH, FK parallel to AB, and meeting EM and CN in P and Q; then $PQ = DB^2$, $CF = PL = AC^2$, and $CH = FM = CD^2$: And $AD^2 + DB^2 = AL + PQ = CF + PL + CH + FM = AC^2 + AC^2 + CD^2 + CD^2$.

Ex. 2. Problem. (The 11th prop. of the 2d book of Euclid.)

To divide a given line, AB, into two parts so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Let a denote the whole $\overline{AB}$

* It may be seen by this construction that $(AC+CD)^2 (AL) = AQ + QL + CK + FN = AC^2 + CD^2 + 2 \times AC \times CD$; and that $(AC-CD)^2 (DB)^2$ or $PQ = PL + QL - HN - KM = AC^2 + CD^2 - 2 \times AC \times CD$.

line

line AB , x the greater of the two parts, and consequently $a-x$ the less; then (by hyp.) $a \times \overline{a-x} = x^2$

$$\therefore x^2 + ax = a^2 \text{ and } x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4}$$

$$\therefore x + \frac{a}{2} = \frac{\sqrt{5a^2}}{2} \text{ and } x = \frac{\sqrt{5a^2} - a}{2};$$

if therefore from a mean proportional between a and $5a$ (found by the 13th prop. of the 6th Book of Euclid) we cut off a line equal to a , and take AH equal to half the remainder, the rectangle under AB and BH will be equal to the square of AH .

140. A quantity not fractional or radical, is said to be of as many dimensions, as there are simple factors in it.

Thus b is of one dimension; ab of two; a^3 of three; $a^3 b^2$ of five.

141. A fractional quantity is said to be of as many dimensions, as the number of simple factors in the numerator exceeds the number in the denominator.

Thus $\frac{ab}{c}$ is of one dimension; $\frac{abc^2d^2}{e^2}$ is of four;

$\frac{ab}{c^2}$ of none.

142. A radical quantity, if it be the square root, is said to be of half as many dimensions, as the quantity

quantity under the sign; if it be the cube root, of one third part as many dimensions; and so on.

Thus $\sqrt{ab}$, $\sqrt[3]{a^2b}$ are of one dimension; $\sqrt{a^2b^2}$ is of two dimensions; $\sqrt[4]{a^{3a-1}b}$ is of three dimensions.

143. If the simple expression for each line be a single letter, then any expression for a line will be of one dimension; any expression for an area, of two dimensions; and any expression for a solid, of three dimensions.

144. Hence any fractional or radical expression for a magnitude, may, either directly or after proper reduction, be distinguished into two factors, one of which is a magnitude of the same dimensions, and the other a fraction of no dimensions.

Thus the line $\frac{ab}{c}$ is equal to $a \times \frac{b}{c}$ the solid $\frac{a^2bcd}{e^2f}$ is equal to $a^3 \times \frac{bcd}{e^2f}$.

The area $\sqrt{a^2b^2}$ is equal to $\frac{\sqrt{a^2b^2} \times \sqrt{a^2b^2}}{\sqrt{a^2b^2}}$,
equal to $\frac{a^2b^2}{\sqrt{a^2b^2}}$, equal to $ab \times \frac{ab}{\sqrt{a^2b^2}}$.

145. All the terms in any equation are of the same number of dimensions, * unless heterogeneous equations have been added together, which is never necessary.

For any primary equations, that state the conditions of a question, being equations between lines, surfaces, or solids, cannot contain terms of different dimensions; and by multiplication, division, involution, or evolution, each term in an equation has its dimensions equally increased, or decreased; so that the terms in any *derived* equation will also be of equal dimensions.

146. If for the magnitudes concerned in any proportion or equation, † proportional numbers be substituted, the resulting numerical equation will be true.

For a ratio compounded of the ratios of magnitudes is the same as a ratio compounded of the

* This consideration is very useful in detecting errors in a complex calculation. Thus, for example, if we find the value of a *line* to be a quantity of two or more dimensions, we may be sure that some mistake has been made; and it is very easy to observe, in passing, whether each term has the right number of dimensions.

† If the magnitudes A, B, C, D, E, F , have a common measure M , and be respectively equal to aM, bM, cM, dM, eM , and fM , (the small letters denoting numbers) A, B, C, D, E and F are said to be as a, b, c, d, e , and f ; now if we have the equation $ABC = DEF$, we shall have $aM \times bM \times cM = dM \times eM \times fM$ or (art. 132) $abc M^3 = def M^3$ $\therefore abc$ will be equal to def .

ratios

ratios of numbers, which are in the proportion of those magnitudes; and the latter ratio is that of the product of the antecedents to the product of the consequents.

147. If therefore in any equation each term be distinguished into two factors, one of which is a geometrical magnitude (of the same dimensions for each term) and proportional numbers be substituted for the other factors, the result will be a true equation between geometrical magnitudes.

Thus if $abcd = defg$, and ab be to cd as 3 to 2, the rectangle $3cd$ will be equal to the rectangle $2fg$.

If the factors, which are geometrical magnitudes, be of the same dimensions as the entire terms themselves, the other factors will be fractions of no dimensions.

Thus in the equation between lines, $\frac{ab}{c} = \frac{d^2}{e} + \frac{g^2}{f}$, if each term be divided into two factors, one of which is a line, the equation becomes $a \times \frac{b}{c} = d \times \frac{d}{e} + g \times \frac{g}{f}$, and the other factors $\frac{b}{c}, \frac{d}{e}, \frac{g}{f}$ are fractions of no dimensions.

In all cases these other factors are each of the same dimensions, and if the terms were so divided,

Q

that

that they were not, the result would be absurd, as being an equation between magnitudes of different dimensions.

148. A fraction of no dimensions may in a peculiar manner be considered as a number; because if its numerical value be substituted for it, the resulting equation will always be legitimate, which is not true of a fraction of any dimensions, unless one of the same dimensions in each term be at the same time converted into a number.

Thus if $\frac{abc}{d} = \frac{efg}{h}$, and e be to h as 3 to 4,

$\frac{abc}{d} = 3 \frac{fg}{4}$; but if e h and f be as 3, 4, and 5,

we cannot with any meaning say that $\frac{abc}{d} =$

$\frac{3 \times 4}{5} g$, i. e. that the *rectangle* which is a 4th proportional to d , a and bc , is equal to 12 5ths of the *line* g .

Moreover, if the numerator and denominator of a fraction of no dimensions be magnitudes, i. e. be of not more than three dimensions, its numerical value may also be considered in this light, viz. as the number denoting how often the denominator is contained in the numerator; for they will be homogeneous magnitudes: Thus $\frac{ab}{cd}$ is the number

number denoting what multiple part or parts ab is of cd . And if the numerator and denominator be of higher dimensions than three, the fraction may be distinguished into factors, each of which admits of the above consideration; thus $\frac{abcd}{efgh}$

($= \frac{ab}{ef} \times \frac{cd}{gh} = \frac{abc}{efg} \times \frac{d}{h} = \&c.$) is the product of the two numbers $\frac{ab}{ef}$ and $\frac{cd}{gh}$, or $\&c.$

Hence it is that fractions of no dimensions are sometimes said to be abstract numbers; and if $A : B :: C : D$, where A and B are supposed to denote two magnitudes of the same kind, and $C : D$ to be any ratio simple or compound, A is put equal to $\frac{C}{D} \cdot B$ where $\frac{C}{D}$ denotes a number.

149. By the preceding articles we are shewn how from the final equation in the solution of a geometrical problem, we may find the numerical value of the magnitude sought, in terms of one or more of the given magnitudes of the same kind.—We may either convert the whole known side into a numerical expression, and so obtain the numerical value of the magnitude sought in terms of any one of the given magnitudes, or we may substitute a number for part only of the known side, leaving the magnitude with which we would

Q 2

compare

compare the unknown quantity, with that number as its co-efficient.

Ex. 1. If $x = \frac{ab}{c}$, and ab and c be to each other as 12, 2, and 3, and we would compare x with b , we may either put $x = \frac{12 \times 2}{3} = 8$, and thence gather that x is to b as 8 to 2 i. e. that $x = 4b$; or substituting only for $\frac{a}{c}$, put $x = \frac{12}{3}b = 4b$.

The former method is the readiest in radical expressions. Thus if $x = \sqrt{ab}$ and a be to b as 4 to 9, putting $x = \sqrt{4 \times 9}$, we find $x = 6$ or $x : b :: 6 : 9 :: 2 : 3$; by the other method $x = \sqrt{ab} = \frac{ab}{\sqrt{ab}} = \frac{a}{\sqrt{ab}} \cdot b = \frac{4}{6} \cdot b = \frac{2}{3}b$.

Ex. 2. If $x = \frac{ab}{c} + \frac{de}{f}$, and a, b, c, d, e, f be as the numbers 1, 2, 3, 4, 5, 6, $x = \frac{2}{3} + \frac{20}{6} = 4$, or $x : b :: 4 : 2 :: 2 : 1$; or substituting only for a, c, d , and f , $x = \frac{1}{3}b + \frac{4}{6}e = (\text{since } e = \frac{5}{2}b)$
 $\frac{1}{3}b + \frac{4}{6} \cdot \frac{5}{2}b = \frac{2}{6}b + \frac{10}{6}b = 2b$.

150. If, in attempting to express one magnitude numerically in terms of another, a surd arises, it shews that the two magnitudes are incommensurable; but at the same time points out the limits of their relative values.

These limits are found by actually extracting the figures of the root indicated.

For if we derive the equation $A = \sqrt[n]{p \times D}$, (A and D being supposed to be two magnitudes of the same kind, and p to be a number, either whole or fractional) and p has an n^{th} root, q , A will be equal to qD $\therefore$ if p has not an n^{th} root, any number r , whose n^{th} power is greater than p would be too great a co-efficient of D; for by hypothesis, $A^n : pD^n$ is a ratio of equality, and p is supposed to be less than r , $\therefore A^n : r^n D^n$ is a ratio of a less to a greater, or A is not so great as rD : and in the same manner it may be shewn, that if s^n be less than p , s is too small a co-efficient of D, or A is greater than sD .

But the nearer r^n or s^n approach to p , the less will $A : rD$ or $A : sD$ differ from a ratio of equality, and the less will A differ from rD or sD . Hence it appears, that by the operation of extracting the n^{th} root of p , we obtain successively, as more and more figures are extracted, defective approximating values for the co-efficient of D.

Thus

Thus in the solution of the problem in art 139 because we find $x = \frac{\sqrt{5a^2}}{2} - a = \frac{\sqrt{5 \times a - a}}{2}$,

and the figures of $\sqrt{5}$ are 2.236, &c. we have the following defective approximating values of x ,

$$\begin{aligned} \text{viz. 1st. } x &= \frac{2a-a}{2} = \frac{a}{2}; \text{ 2d. } x = \frac{2.2 \times a - a}{2} \\ &= \frac{1.2 \times a}{2} = \frac{12a}{2.10} = \frac{6a}{10} = \frac{a}{2} + \frac{a}{10}; \text{ 3d. } x \\ &= \frac{2.23 \times a - a}{2} = \frac{1.23 \times a}{2} = \frac{123a}{2.100} = \frac{100a}{200} \\ &+ \frac{20a}{200} + \frac{3a}{210} = \frac{a}{2} + \frac{a}{10} + \frac{3a}{200}; \end{aligned}$$

&c. &c.

151. These furd's are no defect in the nature of Algebra, but a necessary consequence of the properties of extension: For the only business of Arithmetical operations being the investigation of the numerical value of a quantity in terms of some other quantity of the same kind, if the quantity sought be such, as to have no definite value in terms of that other, i. e. such as to be no multiple part or parts whatsoever of that other, it would be very absurd if by the rules of Algebra we could obtain a value for it.

152. It is common, in order to render a calculation less perplexed, to denote one of the given magnitudes by *unity*. When this is done, the different

different terms of an equation will often appear to be of various dimensions, because unity as a multiplier or divisor, disappears; and any expression will then serve to denote either a number, a line, an area, or a solid. Thus if unity be substituted for a , the lines $\frac{b^2}{a}$, $\frac{bcd^2}{a^2}$, $\frac{a^2}{c}$ will be de-

noted by b^2 , bcd^2 and $\frac{1}{c}$; the expression bc may be

considered either as $\frac{bc}{1^2}$, or $\frac{bc}{1}$, or bc , or $bc \times 1$.

And in order to reduce the terms of an equation in this case to uniformity, we must supply the deficient numerators or denominators with powers of 1; thus the equation $c^2 = b + cd^2$ becomes $c^2 = b \times 1 + \frac{cd^2}{1^2}$, or $\frac{c^2}{1} = b + \frac{cd^2}{1^2}$, or $c^2 \times 1 =$

$b \times 1^2 + \frac{cd^2}{1}$, or &c:

It is to be observed, that in substituting numbers for the other letters, reference must be had to the assumed *unity*; for each magnitude must be expounded by that number, whose ratio to 1 is the same as the ratio of the magnitude to the magnitude *unity*.

153. Further examples of the analytical method of demonstrating geometrical proportions.

Ex,

Ex. 1. Theorem, (being the 47th of the 1st Book of Euclid.)

In any right angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides. (see fig. 2.)

Let ABC be a triangle right angled at A; from A let fall the perpendicular AD on the hypotenuse BC; then (by Euclid prop. 8, B. 6th) the three triangles ABC, ADC and ABC are * similar: Put
AB

* The doctrine of similar triangles is perfectly independent of the 47th of the 1st Book.

The following curious method of demonstrating this proposition, without ratios, is given in Saunderson's Algebra.

Out of eight triangles all similar and equal to the triangle proposed, let there be formed four rightangled parallelograms, AK, BL, CM and DN; (see fig. 3) and let them be disposed as follows: Let EL the longer side of the 2^d. parallelogram be placed in the same straight line with EK the shorter side of the 1st. FM the longer side of the 3^d. in the same straight line with FL the shorter side of the 2^d. and GN the longer side of the 4th. in the same straight line with GM the shorter side of the 3^d.; and let a denote any one of the longer sides, and b any one of the shorter. Then from the uniformity and constitution of the figure, we shall have three squares, viz. ABCD the greatest, EFGH the middlemost, and KLMN the least. It is further evident that the greatest square exceeds the middlemost by four of the triangles above mentioned, and that the middlemost exceeds the least by the other four triangles; and consequently that the greatest and least squares added together are double of the middlemost. But the side of the greatest square is $AB = AE + EB = a + b$; and the side of the least square is $KL = EL - EK = a - b$: Therefore the

area

$AB = a$; $AC = b$; $BC = h$; $BD = c$, and $DC = d$; then since

$h : a :: a : c$, and $h : b :: b : d$, $hc = a^2$ and $hd = b^2$. $\therefore hc + hd = a^2 + b^2$; but $c + d = h$, $\therefore hc + hd = h^2$. $\therefore h^2 = a^2 + b^2$ QED.

Ex. 2. Problem. Upon a given hypotenuse (BC, see fig. 4) to construct a right angled triangle, having an area equal to a given rectangle (b^2 .)

Let h denote the given hypotenuse, x one side, and y the other; then since the rectangle under the sides of a right angled triangle is double the triangle, $xy = 2b^2$; and by the 47th, $x^2 + y^2 = h^2$; from the first equation, $y^2 = \frac{4b^4}{x^2}$. $\therefore$ from the

2^d, $x^2 + \frac{4b^4}{x^2} = h^2$, and $x^4 - h^2 x^2 = -4b^4$,

whence $x^2 = \frac{h^2}{2} \pm \sqrt{\frac{h^4}{4} - 4b^4}$, and $x =$

$\sqrt{\left(\frac{h^2}{2} \pm \sqrt{\frac{h^4}{4} - 4b^4}\right)}$ which put $= a$; also $y^2 =$

$h^2 - x^2 = \frac{h^2}{2} \mp \sqrt{\frac{h^4}{4} - 4b^4}$, and $y =$

$\sqrt{\left(\frac{h^2}{2} \mp \sqrt{\frac{h^4}{4} - 4b^4}\right)}$ which put $= b$.

area of the greatest square is $(a+b)^2 = a^2 + 2ab + b^2$, and the area of the least square is $(a-b)^2 = a^2 - 2ab + b^2$, and their sum is $2a^2 + 2b^2$, half of which, or $a^2 + b^2$, is therefore the area of the middle square; but the middle square is the square on the hypotenuse of the triangle proposed; therefore &c. QED.

If $\therefore$ two circles be described, one with center B and * radius a , and the other with center C and

* There are many different ways of finding a and b , according to the different views, in which the expressions are considered. The method, that will serve most generally, is as follows: $\frac{h^4}{4} - 4b^4$

is composed of the four factors, h, h, h , and $\frac{h}{4} - \frac{4b^4}{h^3}$, of which the

three 1st are given; and the last can be found, for $\frac{4b^4}{h^3}$ is a 3^d propor-

tional to h and $\frac{2b^2}{h}$; $\frac{b^2}{h}$ is to be found by the x th proposition

of the 6th Book of Euclid, and thence, by the same proposition, $\frac{4b^4}{h^3}$ may be found, and being subtracted from $\frac{h}{4}$, we have $\frac{h}{4} - \frac{4b^4}{h^3}$.

which call c , then the rectangle $\sqrt{h^3 \times \frac{h}{4} - \frac{4b^4}{h^3}}$ becomes $\sqrt{h^3 c}$

$= h \sqrt{hc}$; $\sqrt{hc}$ (a mean proportional between h and c) is to be found by the x th of the 6th Book; let it be denoted by d ; then x

$= \frac{h^2}{2} \pm hd = h \times \frac{h}{2} \pm$ and x is had by taking a mean pro-

portional between h and $\frac{h}{2} \pm d$. In a similar manner y may be constructed.

It may be observed that any compound expression of n dimensions may be resolved into n factors, $n-1$ of which are assumed; thus $abc^2 + defg + hb^3$ may be resolved into four factors, three of which

are k, l , and m ; for it is equal to $klm \times \frac{abc^2}{klm} + \frac{defg}{klm} + \frac{hb^3}{klm}$, and

if the magnitudes, which each of the letters denote, be given, each of the three terms in the compound factor may be found by the x th or x th of the 6th Book.

radius

radius b , intersecting each other in A , and the lines BA , CA be drawn, ABC will be the triangle required.

154. The analytical solution of problems in right lined geometry, may always be effected by means of the composition of lines out of parts, and the similarity of triangles; and consequently does not *absolutely require* the application of any other proposition in geometry than the first 46, in the 1st Book of Euclid, the 1st in the 2^d Book, (with the general deduction from it, as explained in art. 138) and the first eight in the 6th Book, together with a few of the succeeding, respecting the finding of 3^d, 4th, and mean proportionals. These propositions, together with the doctrine of ratios, are *sufficient*. But at the same time it is to be observed, that other propositions may often be applied with advantage, both with respect to the brevity and facility of the solution, and to the simplicity and neatness of the construction; especially the 47th of the 1st Book, and the 21st and succeeding propositions in the 3^d Book.

155. It generally happens that a problem requires some scheme to be constructed in order to find the equations necessary for the solution; and the same problem will often admit of many different schemes, all leading to the same conclusion by different ways; and here, as in numerical problems, no general rules can be laid down, that will serve all cases, but the choice must be left to the

skill and experience of the Analyst. Yet the following observations will be found of service.

Having described a diagram, which is supposed to represent truly the quantities, both given and sought, that are concerned in the question, if equations do not readily arise from it, as many in number as there are unknown quantities involved in them, the solution may be promoted by producing some of the lines till they meet, or become of an assigned length; or by drawing from some remarkable point lines parallel or perpendicular to others; or by joining some remarkable points; or, if an angle be given, by compleating the sides into a triangle, either given, or similar to a given one; by resolving an oblique angled triangle into two rightangled triangles; by resolving polygons into triangles; or by *any other construction, by which the scheme may be resolved into triangles, either given, or similar, or rightangled*: For such triangles will always afford a solution. But if we wish to employ some further proposition in geometry, by which we think the solution may be more easily effected, it may perhaps be necessary to accommodate the construction of that particular proposition.

The reader will find abundance of examples to illustrate these remarks in Simpson, Newton, Emerson, Saunderson, and others.

156. The obvious construction of the final equation is often, in point of elegance, far inferior to one, that may be derived from some geometrical proposition, which does not immediately seem to be connected with it: Thus the construction

for the equation $x = \frac{\sqrt{5a^2 - a}}{2}$, in the problem

in art. 139, by finding a mean proportional between a and $5a$, and setting off from the given line, a , half the difference between that mean proportional and a , is not so elegant as the following, derived from the 47th of the 1st Book. (See fig. 5.) From A draw AE perpendicular to AB , and equal to half AB . Join BE , and take AH equal to the difference between EB and EA ; H is the point required,

$$\text{For } EB^2 = AB^2 + AE^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$$

$$\therefore EB = \frac{\sqrt{5a^2}}{2} \text{ and } EB - EA = AH =$$

$$\frac{\sqrt{5a^2 - a}}{2} = x. \text{ This construction agrees with}$$

Euclid's in his 2d book.

157. It happens sometimes that a problem admits of a more elegant solution by constructing for each given condition separately, than any, that can be derived from the analytical investigation.

For

For one condition will perhaps determine that the point, which when found solves the problem, must lie in such a line, and another condition will determine that it must lie in such another line; so that the intersection of these two lines will be the point required.

Thus the problem in art. 153 may be solved as follows. See fig. 6 *First*, on BC describe a parallelogram BCDE, whose area shall be double the proposed area, (by cor. prop. 45 Book 1st.) then by the 41st. and 39th propositions of the 1st Book; the required triangle must lie between the parallels BC and ED; i. e. the vertex of the triangle must lie somewhere in ED or ED produced. *Secondly*, on BC describe a semicircle BAG; and because the angle at the vertex is to be a right angle, and no right angle can be subtended by BC unless it be contained by two lines drawn from B and C to some point in the semicircular arc on BC, the required vertex must also lie in that arc; therefore it must be at A, the intersection of ED and the semicircular arc.

The lines, to which conditions separately determine any particular point, are called the *loci* of that point: Thus the semicircular arc on BC is called the *Locus* of the vertex of *right angled triangles* described in BC as an hypotenuse.

These constructions by the intersections of the *Loci* will always be found the most simple possible:

But

But those derived from an analytical investigation are in general the surest to be obtained.

158. The real difference between an *Algebraical* demonstration, as it is called, and Euclid's *Analytical* demonstrations, is often inaccurately apprehended. The *Algebraical* demonstration is as truly *geometrical* as Euclid's doctrine of ratios is geometrical. It has no *necessary* connexion with numbers, otherwise than that doctrine has : It is not founded on the analogy between products and rectangles, nor does it in any instance require the application of that analogy. It differs from the method Euclid has employed in those propositions, which he demonstrates analytically, in proceeding more closely by the composition and decomposition of ratios, — and in substituting for magnitudes the sums of their parts.

The analogy between products and rectangles is required in finding the numerical value of one magnitude in terms of another ; but the algebraical solution is completed, without any consideration of such values, when it discovers how the magnitudes sought may be constructed ; and the numerical relation between the magnitudes concerned in the final equation, is another, and different deduction from it — one indeed obvious to be made, by reason of the connexion between ratios and the numerical values of magnitudes ; and facilitated by means of the algebraic notation.

This

This analogy may be stated generally as follows: *If numbers be taken that have the same mutual ratio to each other, which certain magnitudes have, i. e. that are the same multiples or parts of each other mutually, as those magnitudes are of each other; then the ratio compounded of the ratios between any number of pairs of those magnitudes, is the same as the ratio between two numbers, the former of which is the product of the numbers proportional to the magnitudes, that are antecedents, and the latter the product of the numbers proportional to the magnitudes that are consequents. In other words, if the numbers a, b, c, d, e, f &c. be proportional to the magnitudes A, B, C, D, E, F , &c. the ratio compounded of the ratios $A : D, B : E, C : F$ &c. is the same as the ratio $abc : def$.* From this we know that in any geometrical equation we may substitute for any expression, that * appears under the form of a product, the real product of numbers proportional to

* The reader must recollect that such an expression as $ab : cd$ is meant primarily to denote the ratio compounded of the ratios $a : c$ and $b : d$; but because rectangles, as appears from Euclid, are to each other in the ratio compounded of the ratio of their sides, ab and cd will also serve to denote the rectangles under the lines a and b , and the lines c and d : And so far numbers do not enter into consideration.—But if $a = 3 M$, and $b = 2 M$, and $c = 5 M$, and $d = 7 M$, then since it has been proved (art. 132 and 118) that the ratio compounded of the ratios $3 M : 5 M$ and $2 M : 7 M$ is equal to the ratio $6 M : 35 M$, it follows that the ratio $ab : cd$ is the same as the ratio $6 M : 35 M$, which ratio leaving out the common magnitude M , whose multiple the numbers denote, is called that of $6 : 35$.

the

the magnitudes denoted by the letters in that expression.

The composition and decomposition of ratios are very greatly assisted by the Algebraic notation; insomuch that without it the prolixity of these analytical demonstrations would be insufferable. If therefore the ancients had not the use of this, or some similar notation, we see why they were compelled to have recourse to another method of demonstration, and at the same time are justified in using the analytical method ourselves. The advantages of it are these, — it is very easy of application, — and it requires but few geometrical principles as a common foundation; the different propositions, that are demonstrated by means of it, being in general independent one of another: That this is an advantage, notwithstanding what has been advanced by some writers concerning the beauty of the connexion between different propositions, the reader will perhaps be inclined to believe, if he considers that the demonstrations of those truths, that are referred to in a proposition, do in fact constitute part of its demonstration, but are omitted to be set down, as being supposed known to the reader: Thus the complete demonstration of the 8th proposition of the 1st book of Euclid requires every word of the demonstrations of the 7th, 5th, and 4th propositions.

So that in estimating the length of a demonstration, consideration ought to be had of all the pre-

vious propositions, whose truth it requires for its establishment. It may further be observed that many propositions are in themselves of little or no value but as steps to some other, from which they are separated merely to prevent the perplexity arising from too great a length of continued demonstration. This is the case, for example, with the 7th of the 1st Book, which is merely subsidiary to the 8th, for if the 8th could be easily demonstrated by means of the 5th and 4th alone it would be useless to retain the 7th. In fact, whoever attempts to demonstrate new propositions, will often find the demonstration so perplexed and tedious, as to be induced to prefix some of the steps as subsidiary propositions or Lemmas. It surely then is desirable to demonstrate each proposition as independently as possible, provided we do not thereby introduce, upon the whole, more subsidiary propositions than would otherwise be necessary. But since there is *probably* an unavoidable necessity of arriving at some truths through the medium of others, art is required in disposing the propositions in the most advantageous and lucid order—In this art Euclid appears to have been a consummate Master, and in this consists the beauty of connexion.

CHAPTER V.

PROPERTIES OF QUADRATIC SURDS.—RULES FOR THE
OPERATIONS WITH SURDS.—OF THE GREATEST
COMMON DIVISORS OF COMPOUND ALGEBRAIC
EXPRESSIONS.

159. **A** *RATIONAL* quantity cannot be equal to the sum or difference of two quadratic surds.

For if a could $= \sqrt{b} \pm \sqrt{c}$ we should have $a - \sqrt{b} = \pm \sqrt{c}$, and $a^2 + b - 2a\sqrt{b} = c$, and $\sqrt{b} = \frac{a^2 + b - c}{2a}$; or $\sqrt{b}$ a rational quantity, contrary to the supposition,

160. *In an equation, each side of which contains a rational part and a quadratic surd, the rational parts are equal to each other, and the surds are equal to each other.*

For if $x + \sqrt{y} = a + \sqrt{b}$, and x were not equal to a , we should have $x - a$, a rational quantity, equal to $\sqrt{b} - \sqrt{y}$; contrary to the last article, $\therefore x = a$, and consequently $\sqrt{y} = \sqrt{b}$,

If a be positive, x is positive; if negative, x is negative; and the same of $\sqrt{y}$ and $\sqrt{b}$. For otherwise the equations $x = a$ and $\sqrt{y} = \sqrt{b}$ could not be true. Hence

161. *The sum of a quadratic surd and rational quantity cannot be equal to the difference of a quadratic surd and rational quantity; nor can a quadratic surd, diminished by a rational quantity, be equal to a rational quantity diminished by a quadratic surd.*

162. *If the product of two quadratic surds be rational, each of the surds is some multiple, part or parts of the other, and $\therefore$ they may be reduced to others, having the same irrational part.*

For if $\sqrt{a} \times \sqrt{b} = n$, $\sqrt{b} = \frac{n}{\sqrt{a}} = \frac{n}{a} \times \sqrt{a}$;

or $\sqrt{b}$ is $\frac{n}{a}$ ths of $\sqrt{a}$, and the surds become

$\sqrt{a}$ and $\frac{n}{a} \sqrt{a}$.

163. Hence, if two quadratic surds be incommensurable with each other, their product is irrational.

164. *If a quadratic surd be the sum or difference of two others, the three surds are commensurable with one another, or may be reduced to surds, having the same irrational part.*

For

For if $\sqrt{a} = \sqrt{b} \pm \sqrt{c}$, $a = b + c \pm 2\sqrt{bc}$, and $a \sim \overline{b+c} = 2\sqrt{bc}$; but $a \sim \overline{b+c}$ is rational, $\therefore 2\sqrt{bc}$ is rational $\therefore$ (art. 162.) $\sqrt{b}$ and $\sqrt{c}$ are commensurable with each other; let $\sqrt{c} = \frac{d}{e} \sqrt{b}$,

then $\sqrt{a} = (\sqrt{b} + \sqrt{c}) = \overline{1 + \frac{d}{e}} \times \sqrt{b}$, or $\sqrt{a}$

and $\sqrt{b}$ are commensurable with each other; and in a similar manner it may be proved that $\sqrt{a}$ and $\sqrt{c}$ are commensurable with each other.

165. If $\overline{a+b}^{\frac{1}{c}} = x+y$, where c denotes an even number, a a rational quantity, b a quadratic surd and x and y , one or both of them, quadratic surds incommensurable with each other, then $\overline{a-b}^{\frac{1}{c}} = x \sim y$.

For raising each side of the equation to the c^{th} power, we have $a+b = x^c + cx^{c-1}y + c \cdot \frac{c-1}{2} x^{c-2}y^2 + \dots + cxy^{c-1} + y^c$: Now the odd terms of this series are rational, because they involve even powers of x and y , and any even power of a quadratic surd is rational; and the even terms are irrational, because if x and y be both surds, and incommensurable with each other, these terms involve the surd xy with rational co-efficients; and if y alone be a surd, they involve the surd y with rational co-efficients: Therefore the sum of the odd terms is a rational quantity, and the sum of the even terms a quadratic surd; therefore

fore (art. 160) a is equal to the sum of the odd terms and b equal to the sum of the even terms

$$\therefore a - b = x^c + c \cdot \frac{c-1}{2} x^{c-2} y^2 \dots + y^c - cx^{c-1} y$$

$$- \&c \dots - cxy^{c-1} = x^c - cx^{c-1} y + c \cdot \frac{c-1}{2}$$

$$x^{c-2} y^2 - \&c. \dots - cx^{c-1} y + y^c = \overline{x-y}^c$$

$$\therefore \overline{a-b}^c = \overline{x-y}^c$$

If a, b, x and y denote *real* quantities, a must be greater than b ; otherwise the equation $\overline{a+b}^{\frac{1}{c}} = x+y$ could not be true; for c being even, $\overline{x-y}^{\frac{1}{c}}$ is positive, whether x or y be the greater term, i. e. the sum of the odd terms in $\overline{x+y}^c$ is greater than the sum of the even terms.

166. If $\overline{a+b}^{\frac{1}{d}} = x+y$, where d denotes an odd number, a and b , one or both, are quadratic surds incommensurable with each other, and x and y are commensurable with a and b respectively, $\overline{a-b}^{\frac{1}{d}} = x-y$.

For, by involution, $a+b = x^d + dx^{d-1} y + \&c.$ where if a and x be rational, the odd terms are rational, and the even terms involve the surd b with rational co-efficients; but if a and x be irrational, the odd terms involve the surd a , and the even terms the surd b : In either case a is equal to the sum of the odd terms, and b to the sum of the even terms, — in the former case by art. 160, and

and in the latter case because else $a \sim$ the sum of the odd terms would be some multiple, part or parts of the surd a , and would be * equal to $b \sim$ the sum of the even terms, which would be some multiple, part or parts of the surd b ; but no multiple, part or parts of a can be equal to any multiple, part or parts of b , because a and b are incommensurable; $\therefore a =$ the sum of the odd terms and $b =$ the sum of the even terms $\therefore a - b = x^d - dx^{d-1}y + \&c \dots - y^d = \overline{x-y}^d$
 $\therefore \overline{a-b}^{\frac{1}{d}} = x-y.$

167. Surds may be denoted either by the radical sign, or by fractional indices. Thus the n^{th} root of a is denoted either by $\sqrt[n]{a}$, or by $a^{\frac{1}{n}}$; the n^{th} root of b^m , by $\sqrt[n]{b^m}$, or by $b^{\frac{m}{n}}$

168. Because a fractional index, $\frac{m}{n}$, signifies indifferently either the n^{th} root of the m^{th} power, or the m^{th} power of the n^{th} root, a quantity with a

* The equation $a \sim$ sum of odd terms $\pm b \sim$ sum of even terms is true at all events; but if a be equal to the sum of the odd terms, each side of the equation is 0, and no absurdity follows. Thus $11\sqrt{2} + 9\sqrt{3} = \sqrt{2+\sqrt{3}}^3 = 2\sqrt{2} + 6\sqrt{3} + 9\sqrt{2} + 3\sqrt{3}$, and $11\sqrt{2} - 2\sqrt{2} - 9\sqrt{2} = 9\sqrt{3} - 6\sqrt{3} - 3\sqrt{3}.$

fractional

fractional index, $\frac{m}{n}$, may be transformed two ways, viz. either to the form of an m power with the index $\frac{1}{n}$, over it, or to the form of an n^{th} root with the index m over it.

$$\text{Thus } \overline{a+x}^{\frac{3}{4}} = \overline{a+x}^{\frac{1}{4}}^3 = \overline{\overline{a+x}^{\frac{1}{4}}^3}^{\frac{m}{n}} = \overline{a^{\frac{1}{n}}^m}^{\frac{1}{n}}.$$

The former of these transformations serves to reduce two surds, whose fractional indices have a common denominator, to surds with a common index: For thus $a^{\frac{m}{n}}$ and $b^{\frac{r}{n}}$ become $\overline{a^m}^{\frac{1}{n}}$ and $\overline{b^r}^{\frac{1}{n}}$.

169. If the denominators of the indices of two surds, or the indices of their radical signs, be the same, the surds are said to be of the same denomination. Thus $\sqrt{a}$, $\sqrt{b^3}$, $a^{\frac{1}{2}}$ and $b^{\frac{3}{2}}$ are of the same denomination; so are $\sqrt[3]{a^2}$, $\sqrt[3]{b}$, $a^{\frac{2}{3}}$ and $b^{\frac{1}{3}}$.

170. A rational quantity may be reduced to the form of a surd, by raising it to the power, which denominates the surd, and then either affixing the radical sign, or a fractional index, or dividing the index by the exponent of the power to

to which the quantity was raised. Thus $a+x$

$$= \sqrt{a+x^2} = \sqrt[3]{a+x^3} = \&c. = \sqrt[4]{a+x^4}^{\frac{1}{2}} = \sqrt[5]{a+x^5}^{\frac{1}{3}} = \&c. = \sqrt[6]{a+x^6}^{\frac{2}{3}} = \sqrt[7]{a+x^7}^{\frac{3}{4}} = \&c. = \sqrt[8]{a^2+2ax+x^2}^{\frac{1}{2}} = \sqrt[9]{a^3+3a^2x+3ax^2+x^3}^{\frac{1}{3}} = \&c.$$

171. A fractional index may have substituted for it any other fraction of equal value—Also the index of the radical sign may be multiplied by any number, if the quantity under the sign be raised to the power denoted by that number, or, which is the same thing, if the index of the quantity under the sign be multiplied by that number, —And conversely, if the index of the radical sign be a composite number, it may be divided by any one of its factors, provided the quantity under the sign has its index divided by the same number, or its root, denominated by that number, extracted.

$$\text{Thus } a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}}; \sqrt{a} \text{ (i. e. } ^2\sqrt{a}) = ^4\sqrt{a^2} = ^6\sqrt{a^3} = \&c; ^{10}\sqrt{a^4} = ^5\sqrt{a^2}; ^6\sqrt{a^5} = ^3\sqrt{a^{\frac{5}{2}}}; ^4\sqrt{a^4-2a^2x^2+x^4} = ^2\sqrt{a^2-x^2}.$$

172. Hence all surds may be reduced to surds of the same denomination—viz. by substituting for their indices indices with a common denominator or by substituting for the indices of their radical signs some common multiple of them all, and raising the quantity under each sign to the power

T

denoted

denoted by the number, into which the index of the sign was multiplied.

Thus $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ = $a^{\frac{2}{4}}$ and $b^{\frac{2}{6}}$; $\sqrt{a}$ and $\sqrt[3]{b}$
 = $\sqrt[6]{a^2}$ and $\sqrt[6]{b^2}$; $\sqrt[4]{a^2-x^2}$ and $\sqrt[6]{b^2+c^2}$ =
 $\sqrt[12]{a^2-x^2)^3}$ and $\sqrt[12]{b^2+c^2)^2}$; $\sqrt[12]{a-x}$ and $\sqrt[8]{y}$
 = $\sqrt[24]{a^2-2ax+x^2}$ and $\sqrt[24]{y^3}$.

173. To multiply two surds together when expressed by fractional indices.

Rule. 1st. If the quantity under the index be the same in both surds, add the indices together for a new index.

2dly. If the quantities be different, reduce the surds (where necessary) to the same denomination by art. 172, then to surds with a common index by art. 168; multiply together the quantities under this common index, and affix the common index to their product.

$$\begin{aligned} \text{Ex. } \sqrt[3]{a+x} \times \sqrt[4]{a-x} &= \sqrt[12]{(a+x)^4} \times \sqrt[12]{(a-x)^3} \\ &= \sqrt[12]{(a+x)^4 \times (a-x)^3}; \quad a^{\frac{1}{2}} \times b^{\frac{1}{3}} = a^{\frac{2}{6}} \times b^{\frac{2}{6}} = \\ &= \sqrt[6]{a^2} \times \sqrt[6]{b^2} = \sqrt[6]{a^2 b^2}; \quad 2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2^{\frac{2}{4}} \times 3^{\frac{2}{4}} = \\ &= \sqrt[4]{2^2 \times 3^2} = \sqrt[4]{4 \times 9} = \sqrt[4]{36}. \end{aligned}$$

The intermediate steps shew the process in the mind, but after some practice the reader will in many cases find it unnecessary to set them down.

174. To multiply surds when expressed by the radical sign.

Rule. Reduce the surds by art. 172, to radicals with a common index; then multiply together the resulting quantities under the signs, and prefix the sign to their product.

$$\begin{aligned} \text{Ex. } \sqrt[3]{a+x} \times \sqrt[4]{a-x} &= \sqrt[12]{a+x}^4 \times \\ \sqrt[12]{a-x}^3 &= \sqrt[12]{a+x}^4 \cdot \sqrt[12]{a-x}^3; \sqrt{a} \times \sqrt{a} = \\ \sqrt[20]{a^5} \times \sqrt[20]{a^2} &= \sqrt[20]{a^5 \times a^2} = \sqrt[20]{a^{7+2}}. \end{aligned}$$

175. To divide one surd by another.

Rule. Reduce as in multiplication of surds, to a common index, and instead of taking the product of the quantities under the common index, take their quotient, to which prefix the common index or radical sign.

$$\begin{aligned} \text{Ex. } \sqrt[3]{a+x} \div \sqrt[3]{a+x} &= \sqrt[6]{a+x}^2 \div \sqrt[6]{a+x}^2 \\ &= \frac{\sqrt[6]{a+x}^2}{\sqrt[6]{a+x}^2} = \sqrt[6]{a+x}^0; \sqrt[n]{a} \div \sqrt[m]{b} = \sqrt[mn]{a^m} \\ \div \sqrt[ms]{b^n} &= \sqrt[ms]{\frac{a^m}{b^n}}; \sqrt[n]{a} \div \sqrt[n]{c} = \sqrt[n]{\frac{a}{c}} \div \\ \sqrt[ms]{\frac{c^n}{a^m}} &= \frac{a^n}{b^n} \div \frac{c^m}{a^m} = \sqrt[ms]{\frac{a^n a^m}{b^n c^m}}. \end{aligned}$$

* The reader must recollect that the index of the square root, though not expressed, is 2.

176. If one of the surds in multiplication or division has a negative index, in order to reduce the surds to the same denomination, either the negative sign must be transferred from the entire fraction to the numerator, or else it must be changed into the positive sign, by converting the quantity under the index into its reciprocal.

$$\text{Ex. } \sqrt[r]{a^{-s}} \times \sqrt[m]{b^s} = \sqrt[r]{a^{-sm}} \times \sqrt[m]{b^{sm}} = \sqrt[r]{a^{-sm} \times b^{sm}},$$

$$\text{or } \sqrt[r]{a^{-s}} \times \sqrt[m]{b^s} = \frac{1}{\sqrt[r]{a^s}} \times \sqrt[m]{b^s} = \sqrt[r]{\frac{1}{a^s} \times b^s}.$$

177. These rules for multiplication and division suppose the quantities to be pure surds without any co-efficients. If we have to multiply or divide surds that have co-efficients, the product or quotient of the co-efficients must be prefixed as a co-efficient to the surd obtained. Thus $a^r \sqrt{b} \times c^s \sqrt{d} = ac^r \sqrt{bd}$; $a^r \sqrt{b} \div c^s \sqrt{d} = \frac{a}{c}^r \sqrt{\frac{b}{d}}$.

It might seem at first sight that in many instances nothing is gained by these rules, the results appearing as complex as if the surds were set down with merely the sign of multiplication or division between them.—What the rules effect is this,—they give the product or quotient in terms of *one* surd alone; thus in Ex. 1st. art. 173 $\sqrt{a+x}^2 \times \sqrt{a-x}^4$ is a single surd; for $\sqrt{a+x}^2$ and $\sqrt{a-x}^4$ are rational quantities. Moreover, in many

many cases they simplify the expressions. But it is often convenient not to employ these rules, especially when the quantities are simple, and fractional indices are used: Thus the product of $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ may often better be denoted by $a^{\frac{1}{m}} b^{\frac{1}{n}}$ then by $a^{\frac{1}{mn}} b^{\frac{1}{mn}}$.

178. If a surd be multiplied or divided by a rational quantity, such quantity may be brought under the sign or index by first reducing it to the form of a surd of the same denomination.

$$\begin{aligned} \text{Thus } a\sqrt{b} &= \sqrt{a^2} \times \sqrt{b} = \sqrt{a^2b}; \quad 2^3\sqrt{a} = \\ &= \sqrt[3]{8a}; \quad \frac{\sqrt[4]{8a}}{2} = \frac{\sqrt[4]{8a}}{\sqrt[4]{16}} = \sqrt[4]{\frac{8a}{16}} = \sqrt[4]{\frac{a}{2}}; \\ 5 \times \sqrt[4]{a-x} &= 5^{\frac{1}{4}} \times \sqrt[4]{a-x} = \sqrt[4]{5^4(a-x)} \end{aligned}$$

179. And conversely, if the quantity under the radical sign, or fractional index with numerator unity, be divisible by a perfect power of the same denomination, that power may be taken out, and its root prefixt as a co-efficient.

$$\begin{aligned} \text{Ex. } \sqrt{a^2b^3} &= ab\sqrt{b}; \quad \sqrt{4a^2-8a^2b} = \\ 2a\sqrt{1-2b}; \quad \frac{\sqrt[3]{a-x}}{8} &= \frac{\sqrt[3]{a-x}}{2^3} = \frac{\sqrt[3]{a-x}}{2} \end{aligned}$$

180. Even when the quantity under the sign or index is not divisible by a perfect power, it may be useful sometimes to divide surds into their component factors, by reversing the operations of multiplying and dividing surds.

$$\begin{aligned} \text{Thus } \sqrt{ab} &= \sqrt{a} \times \sqrt{b}; \quad \sqrt[n]{ab} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}; \\ \sqrt[3]{a^2b-bx^2} &= \sqrt[3]{b} \times \sqrt[3]{a^2-x^2} = \sqrt[3]{b} \times \\ &\sqrt[3]{a+x} \times \sqrt[3]{a-x}; \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}. \end{aligned}$$

181. *In general*, any surd may be transformed to another, having a required co-efficient, by dividing the quantity under the sign by that power of the required co-efficient, which denominates the surd. For let $\sqrt[n]{y}$ represent any surd, and a be the required co-efficient; then $\sqrt[n]{y} = a \sqrt[n]{\frac{y}{a^n}}$.

This article is reduced in idea to art. 179 or 180, by considering the power of the required co-efficient as a component part of the quantity under the radical sign, whether it appears as a factor of it or not. Thus if we consider a as having the factor b^m , viz. as being $\frac{a}{b^m} \times b^m$, we may take

the power b^m out of $\sqrt[n]{a}$ or $\sqrt[n]{\frac{a}{b^m} \cdot b^m}$, and trans-

form the surd to $b^m \sqrt[n]{\frac{a}{b^m}}$.

182. Surds are reduced to lower terms by art. 179, i. e. by extracting the root of some divisor of the quantity under the sign, and if the greatest factor be chosen, that is a perfect power of the required denomination, they are reduced to their lowest terms.

Thus $\sqrt[3]{a^6b^3c}$ is reduced to $a^2\sqrt[3]{b^3c}$; and its lowest terms are $a^2b\sqrt[3]{c}$; the lowest terms of $\sqrt{27}$ are $3\sqrt{3}$.

This reduction is of use for the addition and subtraction of surds. For if two surds, when reduced to their lowest terms, have the same irrational part, their sum or difference is had by taking the sum or difference of their co-efficients for a co-efficient to the common surd; but if they cannot be reduced to the same irrational part, their sum or difference can only be expressed by the signs + or - .

Thus $\sqrt{8} \pm \sqrt{2} = 2\sqrt{2} \pm \sqrt{2} = 2 \pm 1 \times \sqrt{2} = 3\sqrt{2}$ or $\sqrt{2}$. But $\sqrt{7}$ and $\sqrt{2}$ not being reducible to a common irrational part, their sum or difference can be expressed only by $\sqrt{7} \pm \sqrt{2}$.

183. The square root of a binomial, one of whose terms is a quadratic surd, and the other a rational quantity, may sometimes be expressed by a binomial, one or both of whose terms are quadratic surds.

In

In order to know when this may happen let $a \pm \sqrt{b}$ denote the given binomial, and suppose $x \pm y$ its root. Then since $\sqrt{a \pm \sqrt{b}} = x \pm y$, $a \pm b = x^2 + y^2 \pm 2xy$. Now if x and y be one or both quadratic surds, x^2 and y^2 are rational, and $2xy$ must be irrational; assume therefore $x^2 + y^2 = a$, and $2xy = \sqrt{b}$; then $y^2 = \frac{b}{4x^2}$; hence

$$x^2 + \frac{b}{4x^2} = a, \quad x^4 - ax^2 = -\frac{b}{4}$$

$$x^2 = \frac{a \pm \sqrt{a^2 - b}}{2} \quad \text{and} \quad x = \sqrt{\left(\frac{a \pm \sqrt{a^2 - b}}{2} \right)}$$

$$\text{hence } y^2 = (a - x^2) = \frac{a \mp \sqrt{a^2 - b}}{2}$$

$$\text{and } y = \sqrt{\left(\frac{a \mp \sqrt{a^2 - b}}{2} \right)}$$

$$\therefore x \pm y, \text{ the root of } a \pm \sqrt{b} \text{ is } \sqrt{\left(\frac{a \pm \sqrt{a^2 - b}}{2} \right)}$$

$$\pm \sqrt{\left(\frac{a \mp \sqrt{a^2 - b}}{2} \right)}; \text{ a binomial, one or both of}$$

whose terms are simple quadratic surds, provided $\sqrt{a^2 - b}$ be rational; otherwise not. $\therefore$ the square root of a binomial * admits of the proposed form, if

$$* \text{ The square root of any binomial } A+B, \text{ is } \sqrt{\left(\frac{A+\sqrt{A^2-B^2}}{2} \right)}$$

$$+ \sqrt{\left(\frac{A-\sqrt{A^2-B^2}}{2} \right)} \text{ whether one of the terms } A \text{ and } B \text{ be a}$$

surd

if the difference of the squares of the two parts be a square number.

Ex. 1. Req^d the square root of $3 + \sqrt{8}$.

Here $a = 3$, $\sqrt{b} = \sqrt{8}$; $\therefore a^2 - b = 9 - 8 = 1$, a square number, and the root required is

$$\sqrt{\frac{3+1}{2}} + \sqrt{\frac{3-1}{2}} = \sqrt{2} + 1.$$

Ex. 2. Required the square root of $5 - 2\sqrt{6}$.

Here $a = 5$, $\sqrt{b} = 2\sqrt{6}$, and $a^2 - b = 25 - 24 = 1$, and the root required is $\sqrt{\frac{5+1}{2}} - \sqrt{\frac{5-1}{2}}$

$$= \sqrt{3} - \sqrt{2}.$$

OF THE GREATEST COMMON MEASURE OF
COMPOUND ALGEBRAIC EXPRESSIONS.

184. By a rule similar to that in art. 121, vol. 1. may be found the greatest common measure of

surd or not; because whatever A and B be, we may assume $A = x^2 + y^2$, and $B = 2xy$: But unless one of the terms be a surd, these expressions are of no use.

$$\begin{aligned} \text{Ex. } \sqrt{5+4} &= \sqrt{\left(\frac{5+\sqrt{5^2-4^2}}{2}\right) + \sqrt{\left(\frac{5-\sqrt{5^2-4^2}}{2}\right)}} \\ &= \sqrt{\frac{8}{2}} + \sqrt{1} = 2+1. \end{aligned}$$

two Algebraic expressions; * i. e. the most compound expression, that will divide them both without a remainder, so as to give quotients of an integral form.

185. *Lemma.* Let n E and F represent any two compound expressions whatsoever; then, if neither n nor any divisor of n be a divisor of F , any common measure (M) of n E and F is a common measure of E and F .

For M is composed of no divisors not found in F , and therefore has none of the divisors of n ; therefore the divisors of n E , that M contains, are divisors of E ; or M is a common measure of E and F .

186. *Cor.* Hence any common measure of two compound expressions is a common measure of the expressions, which remain after those divisors have been expunged from each, that are not common to both; and the greatest common measure of these remaining expressions is the greatest common measure of the original expressions.

* This greatest common measure is not necessarily the greatest numerical common measure of the numbers resulting from assigning numerical values to the letters concerned in the expressions; e. g. the greatest common measure of $a^2 - c^2$ and $ab + bc$ is $a + c$; but if $a = 7$, $b = 6$ and $c = 3$, the greatest common measure of $49 - 9$ and $42 + 18$ is not $7 + 3$.

187. If nA and B be taken to denote two algebraic quantities, whose greatest common measure is proposed to be found, and n and B be supposed to have no common divisor, then, 1st. if A divides B or mB (m and A having no common divisor) without a remainder, A is the greatest common measure of A and mB and consequently (art. 186) of nA and B . But

2ndly, if mB divided by A gives an integral quotient P with a remainder pC , (p and A having no common divisor) and C divides A without a remainder, C is the greatest common measure of nA and B .

For since C measures A and pC , it measures $PA + pC$ or mB . C then is a common measure of A and mB , therefore of nA and B . But moreover C is the greatest common measure of nA and B . For any common measure of nA and B measures A and PA ; it also measures mB or $PA + pC$, therefore it measures pC ; but measuring A and pC , it measures C ; therefore no common measure of nA and B has factors not contained in C .

3dly. Making the same suppositions concerning the small letters, * viz. that such as are co-efficients

* The small letters make the investigation general; such of them as in any particular example are not wanted, may then be considered as denoting unity.

ents in the remainders have no factors in common with the preceding divisors, and that such as are co-efficients in the dividends have no factor in common with the divisors to those dividends, in exactly the same manner it may be proved, that if mB divided by A gives a quotient P with a remainder pC , and qA divided by C gives a quotient R with a remainder rD , and D divides C without a remainder, D is the greatest common measure of nA and B .

In general, whatever be the number of divisions, the last divisor will be the greatest common measure required.

From these considerations we obtain the following :

188. *Rule for finding the greatest common measure of two algebraic quantities.*

Expunge from the less complex of the two given quantities any factor, that appears upon inspection, and is not common to both; by the remaining part divide the other given quantity, or that other multiplied into any quantity, not a factor of the divisor, that may be necessary to make the first term of the quotient integral, so far as integral-quotient terms can be obtained. Expunge from the remainder any factor, that appears upon inspection, and is not common to the last divisor; by the remaining part divide the last divisor, or that last divisor multiplied into any

any quantity, not a factor of the new divisor, that may be necessary to make the 1st. term of the quotient integral, so far as integral quotient terms can be obtained. Proceed thus till there be no remainder—The last divisor is the greatest common measure required.

Ex. 1. Required the greatest common measure of $8a^2b^2 - 10ab^3 + 2b^4$ and $9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4$.

The former of these expressions contains the factor $2b^2$ or $2b \times b$, of which the part b only is common to both expressions; dividing therefore the former by $2b$, the quotient, viz. $4a^2b - 5ab^2 + b^3$, is to be the divisor. And in order that the 1st. term of the quotient may be integral, the other expression must be multiplied by 4. Then the work will proceed as follows :

$$\begin{array}{r}
 4a^2b - 5ab^2 + b^3 \quad 36a^4b - 36a^3b^2 + 12a^2b^3 - 12ab^4 \quad (9a^2 \\
 \underline{36a^4b - 45a^3b^2 + 9a^2b^3} \\
 \text{Remr. } 9a^3b^2 + 3a^2b^3 - 12ab^4
 \end{array}$$

This remainder contains the factor $3ab^2$ or $3ab \times b$, of which the part b only is common to the last divisor—It must be divided therefore by $3ab$ to obtain a new divisor; and to make the division proceed, the last divisor must be multiplied by 3 for a new dividend. Then the work will proceed as follows.

$$\begin{array}{r}
 3a^2b + ab^2 - 4b^3 \quad 12a^2b - 15ab^2 + 3b^3 \quad (4 \\
 \underline{12a^2b + 4ab^2 - 16b^3} \\
 -19ab^2 + 19b^3
 \end{array}$$

This

This remainder contains the factor $-19b^2$ or $-19b \times b$, of which the part b only is common to the last divisor; therefore this remainder must be divided by $19b$; and $3a^2b + ab^2 - 4b^3$ being made the dividend, the work will proceed as follows

$$\begin{array}{r} ab - b^2 \overline{) 3a^2b + ab^2 - 4b^3} \quad (3a + 4b \\ \underline{3a^2b - 3ab^2} \end{array}$$

$$4ab^2 - 4b^3$$

$$\underline{4ab^2 - 4b^3}$$

There being no remainder, $ab - b^2$ is the greatest common measure of the two given quantities.

Ex. 2. Required the greatest common measure of $2a^3 + ba - b^2$ and $a^3 + ba^2 - a - b$.

$$\begin{array}{r} 2a^3 + ba - b^2 \overline{) 2a^3 + 2ba^2 - 2a - 2b} \quad (2a^2 + ba^2 - b^2a \\ \underline{2a^3 + ba^2 - b^2a} \end{array}$$

$$ba^2 + b^2a - 2a - 2b$$

$$\begin{array}{r} ba^2 + b^2a - 2a - 2b \overline{) 2ba^2 + b^2a - b^3} \quad (2 \\ \underline{2ba^2 + 2b^2a - 4a - 4b} \end{array}$$

$$\underline{4a - b^2a + 4b - b^3}$$

* Or $+19b^2$. If $+19b^2$ be employed, the greatest common measure will come out $b^2 - ab$ instead of $ab - b^2$.

Dividing

Dividing this remainder by $4 - b^2$, we have $a + b$ for a new divisor.

$$\begin{array}{r} a+b \overline{)ba^2 + b^2a - 2a - 2b(ba - 2)} \\ \underline{ba^2 + b^2a} \end{array}$$

$$\begin{array}{r} -2a - 2b \\ \underline{-2a - 2b} \end{array}$$

The greatest common measure is $a + b$.

CHAPTER VI.

PROPERTIES OF NUMBERS. LOGARITHMS.

189. *THE least common multiple of two numbers (a and b) is equal to their product divided by their greatest common measure (c .)*

For 1st. their least common multiple must contain as a divisor, the product of all the different prime factors in a and b ; and if in either of them any factor be repeated, it must be as often repeated in their least common multiple: *e. g.* if d, e, f, m, n be prime numbers, the least common multiple of def and d^3mn must contain $defmn$ as a divisor: Because it must be divisible by any one of the factors d, e, f, m, n , and by the doctrine of prime numbers (see incommensurables) no number except $defmn$ or some multiple of it is so divisible. In like manner the least common multiple of d^2e^2f and d^3mn must contain as a divisor d^3e^2fmn .

But 2^{dly}. it is evident that the least common multiple requires no other factor besides this product, because this product is divisible by either of the given numbers. Thus $defmn$ is divisible by
def

def or by $demn$; $d^3 e^2 fmn$ is divisible by $d^2 e^2 f$ or by $d^3 mn$.

Since then the least common multiple does require this product as a divisor, but requires nothing more, this product is the least common multiple.

Now this product of the different factors in a and b is equal to $\frac{ab}{c}$: For if $a = cr$ and $b = cs$, this product is $a \times s$ or $b \times r$ or crs ; and $crs = \frac{cr \times cs}{c} = \frac{ab}{c}$: Thus in the first of the above

examples $c = de$, and $defmn = \frac{def \times demn}{de}$; in the

second $c = d^2$ and $d^3 e^2 fmn = \frac{d^2 e^2 f \times d^3 mn}{d^2}$

190. Cor. Any other common multiple of two numbers is a multiple of their least common multiple.

191. In a similar manner it may be proved that if m be the least common multiple of a and b , the least common multiple of m and c is the least common multiple of a, b and c ;—that if m be the least common multiple of a and b , and n of m and c , the least common multiple of n and d is the least common multiple of a, b, c and d ;—in general,

162 PROPERTIES OF NUMBERS.

192. *The least common multiple of any number of quantities is the least common multiple of these two, viz. the last quantity, and the least common multiple of all the preceding.*

For let M be the least common multiple of any number of the quantities a, b, c, d , &c. then M is the product of all the different factors in a, b, c , &c. therefore M is had by multiplying a into the factors of b not common to a , multiplying the result (m , or the least common multiple of a and b) into the factors of c not common to m , multiplying the result (n , or the least common multiple of m and c) into the factors of d not common to n , and so on.

In small numbers, which can easily be distinguished into their component parts, the easiest * method

* The following practical rule is given by Mr. Bonycastle in his useful treatise of Arithmetic, Edition 5th. page 101.

"1. Divide by any number that will divide two or more of the given numbers without a remainder, and set the quotients, together with the undivided numbers, in a line below them.

"2. Divide the second line as before, and so on till there are no two numbers that can be divided; then the continued product of the divisors and quotients will give the multiple required.

Ex. What is the least common multiple of 3, 5, 8 and 10?

$$\begin{array}{r} 5 \overline{) 3, 5, 8, 10} \\ 2 \overline{) 3, 1, 8, 2} \\ 3, 1, 4, 1 \end{array}$$

$$5 \times 2 \times 3 \times 4 = 120 \text{ the answer.}"$$

The

method of actually finding the least common multiple is to take by inspection all the different factors and multiply them together,

X 2

Ex,

The divisors in this rule ought to be restricted to prime numbers — then it may be demonstrated as follows:

Let p, q, r be the given numbers, and let the work stand thus,

$$\begin{array}{r} a)p, q, r \\ \hline a)p, q, r \\ \hline p, q, r \end{array}$$

a and a being prime numbers, and p, q, r having no common factor; then I say $a a p q r$ is the least common multiple of p, q and r .

For 1st. after the given quantities, as many of them as contain the prime factor a , have been divided by a , the prime factor a remains among them; therefore $a a$ must be a factor of any common multiple of them; This is evident if a and a be different numbers, and if they be the same, some one of the given quantities must have been divisible by a , and then by a again, i. e. must have contained a^2 or $a a$.

2^{dly}. There can be no number but p or some multiple of p , which multiplied into $a a$ can produce a number containing the factor p ; — for p is either the factor left after the prime factors a and a have both been taken out of p , or after one of them has been taken but, or 'tis p itself; according as the a 's are both, one, or neither of them divisors of the p 's; — therefore $a a p$ is a factor of any common multiple of the given quantities: In the same manner it may be shewn that $a a q$ is a factor; that $a a r$ is a factor. Therefore since

$p a$

164 PROPERTIES OF NUMBERS.

Ex. The least common multiple of 12, 18 and 20 or $2^2 \times 3$, 2×3^2 and $2^2 \times 5$ is $2^2 \times 3^2 \times 5$ or 180;

The least common multiple of the nine digits is $9 \times 8 \times 7 \times 5$.

193. *The decimal value of a fraction is interminate if the denominator of the fraction, when reduced to its lowest terms, contains any other factor than 2 or 5.*

¹_p, ¹_q, ¹_r contain no common factor, ¹_a ¹_a ¹_p ¹_q ¹_r must be a factor of any common multiple of the given quantities; i. e. their least common multiple is either ¹_a ¹_a ¹_p ¹_q ¹_r or requires some factor besides; but it does not require any other, because ¹_a ¹_a ¹_p, ¹_a ¹_a ¹_q, ¹_a ¹_a ¹_r are respectively equal to or else multiples of ¹_p, ¹_q and ¹_r. Therefore ¹_a ¹_a ¹_p ¹_q ¹_r is the least common multiple of the given quantities.

In the same manner the rule may be demonstrated whatever be the number of given quantities and of divisions.

If ¹_a and ¹_a be not prime numbers, it may happen that ¹_a ¹_a ¹_p has some factor unnecessarily repeated; for example ¹_a may not be a divisor of ¹_p, and yet some factor of ¹_a may be; so that ¹_p being taken down into the third line shall contain a factor not wanted in addition to ¹_a ¹_a. Thus

$5 \times 2 \times 3 \times 2 \times 5 \times 7$ is not the least common multiple of 50, 30 and 42 it containing an unnecessary factor 5.

$$\begin{array}{r} 5) 2 \times 5 \times 5, 2 \times 3 \times 5, 2 \times 3 \times 7 \\ 2 \times 3 \overline{) 2 \times 5, 2 \times 3, 2 \times 2 \times 7} \\ \underline{2 \times 5, 1, 7} \end{array}$$

For

For the numerator and denominator of the reduced fraction have no common factor, and the addition of cyphers to the right of the numerator, which is multiplying it by some power of 10, introduces no other prime factor than 2 and 5, so that if the denominator contains any other prime factor than 2 or 5, the numerator with cyphers added to it will not be divisible by it without a remainder; because one number will not divide another without a remainder, unless that other contains all the factors of the divisor.

Hence the fractions $\frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$ &c,

$\frac{2}{3}, \frac{2}{7}, \frac{2}{9},$ &c, $\frac{3}{7}, \frac{3}{11},$ &c. &c. are not reducible to terminating decimals.

194. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

$$\text{Or } \overline{a+b} \times \overline{a-b} = a^2 - b^2.$$

If $a = b + 1$, $\overline{a+b} \times \overline{a-b} = a + b$; hence

195. *Cor. The sum of two numbers that differ by unity, is equal to the difference of their squares.*

196. *The sum of any number (n) of terms in the series of odd numbers, 1, 3, 5, 7, &c. is equal to (n²) the square of that number.*

For

For, by the theorem for arithmetical series, viz.

$$s = \overline{2a + n-1.b} \times \frac{n}{2}, \text{ the sum of } n \text{ terms of}$$

the series 1, 3, 5, 7, &c. is $\overline{2 + n-1.2} \times \frac{n}{2},$

$$= \frac{2n}{2} + \frac{2n^2}{2} - \frac{2n}{2} = n^2.$$

197. Cor. *The differences between the contiguous terms in the series $1^2, 2^2, 3^2, 4^2,$ &c. form the series of odd numbers 3, 5, 7, &c.*

$$\begin{aligned} \text{For } 2^2 - 1^2 &= \overline{1+3} - 1 = 3; 3^2 - 2^2 = \\ \overline{1+3+5} - \overline{1+3} &= 5; 4^2 - 3^2 = \overline{1+3+5+7} \\ - \overline{1+3+5} &= 7; \text{ and so on.} \end{aligned}$$

This article would easily follow as a cor. to art. 194.

198. *The product of any two contiguous numbers in the natural series 1, 2, 3, 4, 5, &c. is divisible by 1×2 ; of any three, by $1 \times 2 \times 3$; of any four, by $1 \times 2 \times 3 \times 4$; and so on.*

The method of proving this will easily appear from any particular example. Let a, b, c, d, e, f, g, h denote any eight contiguous numbers; then

1st. It is evident that one of any two contiguous number is a multiple of 2; one of any three, a multiple of 3; and so on. Therefore $abcdefgh$ is at least divisible by the product of all the prime factors

factors in 1,2,3,4,5,6,7,8, viz. by $1 \times 2 \times 3 \times 5 \times 7$. And it only remains to be shewn that 2dly, It contains a sufficient repetition of the factors 2 and 3 to be divisible by the whole of $1 \times 2 \times 3 \times 4 \dots \times 8$. Now each of the four numbers ab, cd, ef, gh is divisible by 2; one of the four a,b,c,d , and one of the four e,f,g,h are divisible by 4; and one of the eight is divisible by 8; $\therefore abcdefgh$ is divisible by $2 \times 4 \times 8 \times 2$.

Again abc and def each contain the factor 3, $\therefore abcdef$ is divisible by 3^2 . Hence $abcdefgh$ is divisible by $2^2 \times 3^2$, i. e. by $2 \times 3 \times 4 \times 6 \times 8$; it is also divisible by 5×7 , $\therefore$ it is divisible by $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$.

Aliter.

The product of any two contiguous numbers $p, p-1$ divided by 2 is the number of combinations in p things taken two and two together, and is therefore a whole number; or $p \times p-1$ is divisible by 2.

The product of any three contiguous numbers $p, p-1, p-2$ divided by 2×3 is the number of combinations in p things taken three and three together, and is therefore a whole number; or $p \times p-1 \times p-2$ is divisible by 2×3 . And so on. See combinations, Vol. I.

199. Cor. If n be a prime number greater than 2, $n \cdot \frac{n-1}{2}$, and $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, and &c. are each divisible by n without a remainder.

For since $n \cdot \overline{n-1}$ is divisible by 2, and n does not contain the factor 2, $\overline{n-1}$ must contain it; $\therefore \frac{n-1}{2} \left(= n \cdot \frac{n-1}{2} \div n \right)$ is a whole number.

And so of the others.

200. If n be a prime number, and any number and its n^{th} power be each divided by n , the two remainders will be the same.*

For 1st 1^n divided by n leaves a remainder 1.

$$2^{\text{ndly}} 2^n = \overline{1+1}^n = 1 + n + n \cdot \frac{n-1}{2} + \&c \dots + 1$$

divided by n leaves the same remainder as $1+1$ divided by n , because the intermediate terms are divisible by n without a remainder (art. 199.)

$3^{\text{dly}} 3^n = \overline{2+1}^n = 2^n + n \times 2^{n-1} + \&c. + 1$ divided by n leaves the same remainder as $2^n + 1$ divided by n , viz. one more than 2^n divided by n , $\therefore$ one more than 2 divided by n ; $\therefore$ the same remainder as 3 divided by n .

* When the number to be divided is less than the divisor, by the remainder is meant the number itself. Thus the remainder of 2 divided by 5 is 2.

4^{th} $4^n = 3 + 1^n = 3^n + n \times 3^{n-1} + \&c. + 1$ divided by n leaves the same remainder as $3^n + 1$ divided by n , viz. *one* more than 3^n divided by n , $\therefore$ *one* more than 3 divided by n ; $\therefore$ the same remainder as 4 divided by n .

And thus we may proceed, the remainders increasing by unity, till we come to n and n^n divided by n , when the remainders will be 0: for if it be true of one number, it is true of the next: $\therefore$ any number less than n and its n^{th} power divided by n , leave the same remainder, viz. the number itself, and in the same manner it may be demonstrated of all numbers greater than n , but more easily and generally thus.

Let $pn+q$ denote any number greater than n , q being supposed less than n ; then $pn+q$ divided by n leaves a remainder q , and $\overline{pn+q}^n$ divided by n viz. $\overline{pn}^n + n \times \overline{pn}^{n-1} q + \&c. + npnq^{n-1} + q^n$ divided by n leaves the same remainder as q^n divided by n , because all the previous terms are divisible by n without a remainder; but by what has been already shewn, q^n divided by n leaves a remainder q .

Therefore $\overline{pn+q}$ and $\overline{pn+q}^n$ divided by n leave the same remainder.*

Y

Therefore

* After I had discovered this property and its demonstration, I found that Dr. Waring had before given it in his *Meditationes Algebraicae*;

201. If n be any prime number greater than 2, and any number and its n^{th} power be each divided by $2n$, the two remainders will be the same.

For let $2^n = pn + 2$, then

1st. since 2^n is even, $pn + 2$ is even, $\therefore pn$ is even
 $\therefore$ since n is odd, p is even, or contains the factor 2, $\therefore pn$ is divisible by $2n$, $\therefore 2^n$ or $pn + 2$ divided by $2n$ leaves the same remainder as 2 divided by 2^n . Let $p = 2b$.

2^{dly}. $3^n = \overbrace{2+1}^n = 2^n + n, 2^{n-1} + \&c. + 1$, the intermediate terms of which being divisible by $2n$, the whole divided by $2n$ leaves the same remainder as $2^n + 1$ divided by $2n$, or (by the first head) the same remainder as 3 divided by $2n$. Let $3^n = 2cn + 3$.

3^{dly}. $4^n = 2^n \times 2^n = \overbrace{2bn+2} \times \overbrace{2bn+2} = \overbrace{2bn}^2 + 2 \times 4bn + 4$, which divided by $2n$ leaves the same remainder as 4 divided by $2n$.

And thus we may proceed, the remainder increasing by unity, till we come to $2n$ and $\overbrace{2n}^n$ divided by $2n$, when the remainders will be 0. For

Algebraic; but the nature of the Professors demonstration was too difficult and unaccommodated to this Work to induce me to alter any thing in my own.

I do not know whether the property in the next article has hitherto been noticed by any writer.

if

if it be true of two numbers; it is true of their product; $\therefore$ it is true of any even number, if it be true of all the preceding numbers—and if it be true of an even number, it is true of the next odd number.

Therefore if the n^{th} power of any number less than $2n$ and the number itself be each divided by $2n$, the remainders will be the same, viz. the number itself.

For all numbers greater than $2n$ the article may be demonstrated generally as follows.

Let $2nr + q$ denote any number greater than $2n$, q being supposed less than $2n$, then $2nr + q$ divided by $2n$ leaves a remainder q , and $\overline{2nr + q}^n = \overline{2n}^n + n \cdot \overline{2nr}^{n-1} q + \&c. + q^n$ divided by $2n$ leaves the same remainder as q^n divided by $2n$ but it has been already shewn that q^n divided by $2n$ leaves a remainder q , $\therefore 2nr + q$ and $\overline{2nr + q}^n$ divided by $2n$ leave the same remainder.

202. *If unity be subtracted from any integral power of any number, the remainder is divisible by the next inferior number. Or, if n denotes a whole number, $r^n - 1$ is divisible by $r - 1$.*

The quotient is $r^{n-1} + r^{n-2} + \&c. \dots + 1$

Thus $10 - 1$, $10^2 - 1$, $10^3 - 1$, $\&c.$ are divisible by 9, the quotients being 1, 11, 111, $\&c.$

203. *If from any number the sum of its digits be subtracted, the remainder is divisible by 9.*

For if a, b, c &c. be the digits of the number in their order, beginning from the units, the number is $a + 10b + 10^2 c + 10^3 d + \&c$; from which if $a + b + c + d + \&c$ be subtracted, the remainder is $\overline{10-1} . b + \overline{10^2-1} . c + \overline{10^3-1} . d + \&c$. which, by the last art., is divisible by 9.

204. Cor. *If any number and the sum of its digits be each divided by 9, the remainders will be the same.*

For if S be the sum of the digits, the number, agreeably to the above notation, is

$S + \overline{10-1} . b + \overline{10^2-1} . c + \&c$; which, divided by 9, leaves the same remainder as S divided by 9, because all the terms except the first are divisible by 9 without a remainder.

205. *The two last articles are evidently true of 3, as well as 9, 3 being an aliquot part of 9.*

206. These properties belong to 9 and 3 on account of 10 being the root of the scale, or that number, according to whose powers the value of any digit increases as it is removed higher and higher to the left. In a scale, whose root was 9, they would belong to 8, and its aliquot parts—In a scale, whose root was 11, to 10, and its aliquot parts.—In general, if r be the root of the scale, these

these properties belong to $r-1$, and its aliquot parts: This readily appears if r and its powers be inserted instead of 10 and its powers in the foregoing demonstrations.

When r denotes the root of the scale, and a, b, c , &c. the digits, the number is $a + br + cr^2 + dr^3 + \&c.$ Hence to express a number according to the scale, whole root is r , we must find what is the highest power of r contained in that number; and seek how often that power is contained in it; the answer is the first or highest digit in the required expression—Then, subtracting from the given number *the value* of this digit, viz. *the digit multiplied into that highest power of r* , we must seek how often the next inferior power of r is contained in the remainder; the answer is the second digit. And so on.

Ex. Let $r = 4$, and the number to be expressed be *one hundred and ninety-seven*.

The highest power of 4 contained in this number is 4^3 which is contained in it 3 times, with a remainder five; $\therefore$ 3 is the first digit—next, five contains 4^2 not once, $\therefore$ 0 is the second figure—again, five contains 4 once, with a remainder 1; $\therefore$ 1 is the third figure, and there remains 1 for the place of units. So that the above number must be expressed by 3011, if the value of each digit increases four-fold by being removed one place to the left.

The

The digits in any example may be easily determined by the following.

Rule. Express the given number according to the decimal scale, and divide it by r , the root of the proposed scale—set by the remainder, which is the number of units the given number contains, besides powers of r .—Divide the quotient by r again—set by the remainder, which is the number of r 's the given number contains, besides higher powers of r .—Divide the last quotient by r again, and proceed as before, till the quotient be no longer divisible by r .—The last quotient, together with the several remainders in their order from last to first, form the required expression.

Ex. What is the expression for the number *one hundred and ninety seven*, when $r = 4$

$$\begin{array}{r} 4 \overline{) 197} (1 \\ 4 \overline{) 49} (1 \\ 4 \overline{) 12} (0 \\ \underline{3} \\ 3011 \end{array}$$

Answer

LOGAR

LOGARITHMS.

207. *Logarithms are the measures of ratios.* See "ratios considered as quantity." Vol. 1st. from art. 291, to the end of the chapter.

Any proposed ratio having a magnitude assumed at pleasure for its Logarithm, the Logarithm of any other ratio will be that multiple, part or parts of the assumed logarithm, that this other ratio is of the proposed ratio; and if the logarithm of any ratio be taken negatively, it is the logarithm of the * inverse of that ratio.

Thus, if we assume 1 the measure of the ratio $16 : 1$, we shall have

2 the measure of the ratio $\sqrt{16}^2 : 1$ or $256 : 1$

$\frac{1}{2}$ — — — — — $4 : 1$

$\frac{1}{4}$ — — — — — $2 : 1$

$\frac{3}{4}$ — — — — — $8 : 1$

$\frac{5}{4}$ — — — — — $32 : 1$

—1 — — — — — $1 : 16$ or $\frac{1}{16} : 1$

—2 — — — — — $1 : \sqrt{16}^2$ or $\frac{1}{16^2} : 1$

* See art. 294,—Vol. I.

208. It is convenient to reduce all ratios to such terms as have the common consequent *unity*, —which is done by dividing both the terms by the consequent; for thus the ratio $a : b$ becomes $\frac{a}{b} : 1$

209. The logarithm of a ratio, whose consequent is *unity*, is usually called the logarithm of the antecedent. Thus the logarithm of the ratio $3 : 1$ is called the logarithm of 3.

210. A table of common logarithms is a set of these numerical measures, ready computed, for all numbers within certain limits, and in which 1 is the logarithm of the ratio $10 : 1$ or of the number 10.

211. The use of such tables is to abridge the labor of calculations by enabling us to substitute for multiplication, division, involution and evolution the respectively more easy operations of addition, subtraction, multiplication and division.

This abridgement, which belongs to any other system of logarithms as well as that where the logarithm of 10 is 1, is founded on the following principle, viz. that the sum of the logarithms of two numbers is equal to the logarithm of their product—and the difference of the logarithms of two numbers to the logarithm of quotient of the

the

the greater number divided by the less; and that difference taken negatively, to the logarithm of the quotient of the less divided by the greater; or, generally, that if from the logarithm of one number be subtracted the logarithm of another, the result will be the logarithm of the quotient of the former number divided by the latter.

The truth of this principle is easily shewn, for logarithms being the measures of ratios, and being positive or negative according as the ratios are positive or negative, the sum of the logarithms of two ratios is the logarithm of that ratio which is the sum of those two ratios; and if one logarithm be subtracted from another, the result* is the logarithm of that ratio, which results from subtracting the former ratio from the latter. Now the sum of the ratios $a : 1$ and $b : 1$ is the ratio $ab : 1$; and the ratio had by subtracting the ratio $b : 1$ from the ratio $a : 1$ is $\frac{a}{b} : 1$; $\therefore \log.$

$$\log. a : 1 + \log. b : 1 = \log. ab : 1 \text{ and } \log.$$

$$\log. a : 1 - \log. b : 1 = \log. \frac{a}{b} : 1; \text{ or, according}$$

* As the word *subtraction* is here used in its most extended sense, generally to call this result the remainder would in some cases be inconsistent with the usual meaning of words. Thus if -4 be subtracted from 1 , the result, 5 , is not properly a remainder.

to the concise mode of speaking, (see art. 209) $\log. a + \log. b = \log. ab$, and $\log. a - \log. b = \log. \frac{a}{b}$

Hence the logarithm of any product is equal to the sum of the logarithms of the factors; for $\log. abc = \log. ab + \log. c = \log. a + \log. b + \log. c$; and so on; and the logarithms of powers and roots are those multiples and sub-multiples of the quantity under the index, which the index denotes; for $\log. a^2 = \log. a + \log. a = 2 \times \log. a$; $\log. a^3 = \log. a^2 + \log. a = 3 \times \log. a$; and so on; and conversely $\log. a^{\frac{1}{2}} = \frac{1}{2} \times \log. a$, $\log. a^{\frac{1}{3}} = \frac{1}{3} \times \log. a$; and so on.

Having therefore a table of logarithms, we may reduce the labor of multiplication and division to the much easier operations of addition and subtraction—that of involution to a simple multiplication—and that of evolution to a simple division; for instead of multiplying two numbers together to find their product we may add together their logarithms, and find what number has the sum for its logarithm—that number is the product; and the like in other cases.

In the common system, where the logarithm of 10 : 1 is 1, it is plain that the logarithm of the double of the ratio 10 : 1, viz. the ratio 100 : 1, is 2; of the triple of 10 : 1, viz. 1000 : 1, 3; and so

so on; that the logarithm of $1 : 10$ or $.1 : 1$ is -1 ; of $1 : 100$ or $.01 : 1$, -2 ; and so on: that the logarithm of any ratio greater than the ratio $10 : 1$ but less than $100 : 1$, as, for example, the ratio $50 : 1$, is greater than 1 , but less than 2 . &c. &c. Or according to the conciser mode of speaking, that the logarithms of $100, 1000$, &c. of $.1, .01$, &c. are $2, 3$, &c. $-1, -2$, &c. and that the logarithm of any number between 10 and 100 is between 1 and 2 ; of any number between 100 and 1000 , between 2 and 3 ; &c. &c.

212. It is evident that a table of logarithms might be defined to be a set of numbers so adapted to others, that the sum of any two of them shall belong to the product of the two numbers to which the two belong.

213. The indices of the common ratio (r) in a geometrical progression, beginning from unity, have this property with respect to the terms of the progression.

For in the series $1, r^1, r^2, r^3$, &c. if we take any two terms, as for example the 2^d and 4^{th} and add together the indices of r in those terms, their sum (4) will be the index of r in that term (r^4) which is the product of these two terms.

214. Hence the indices of the common ratio in such a geometrical progression are logarithms of

the ratios which the terms bear to unity, or, according to the conciser mode of speaking, the logarithms of the terms themselves.

Thus 0 is the logarithm of the ratio 1 : 1 or $r^0 : 1$, or of 1 : 1 of $r^1 : 1$ or of r ; 2 of $r^2 : 1$ or of r^2 ; if the series be continued backwards through the terms r^{-1} , r^{-2} , &c. we have - 1 the logarithm of $r^{-1} : 1$ or of r^{-1} ; - 2 of $r^{-2} : 1$ or of r^{-2} ; &c. &c.

215. This series gives the logarithms of none of the numbers intermediate between 1 and r , r and r^2 , &c; but by inserting mean proportionals between the terms, the progression and its indices preserve their properties, and exhibit the logarithms of a greater variety of numbers. Thus by inserting *one* mean proportional between each two adjoining terms, we have the geometrical progression 1, $r^{\frac{1}{2}}$, r^1 , $r^{\frac{3}{2}}$, r^2 , $r^{\frac{5}{2}}$, r^3 , &c; and 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, &c. the logarithms of 1, $r^{\frac{1}{2}}$, r , $r^{\frac{3}{2}}$, r^2 , $r^{\frac{5}{2}}$, r^3 &c. respectively—By inserting *two* mean proportionals, we have the geometrical series 1, $r^{\frac{1}{3}}$, $r^{\frac{2}{3}}$, r^1 , $r^{\frac{4}{3}}$, $r^{\frac{5}{3}}$, r^2 , &c. and 0, $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, &c. the logarithms of the respective terms.

216. And since by increasing the number of these mean proportionals the difference between any two contiguous terms decreases sine limite, it follows

lows that if r be any number whatsoever greater than unity, we may so increase the number as to make some one or other of the terms agree as nearly as we please, with any proposed number — so that though the natural numbers 1, 2, 3, 4, &c. cannot by any means be introduced into a geometrical series, approximations to them may, which will serve for practical purposes. Taking therefore $\frac{1}{n}$ of a proper minuteness to obtain the

proposed degree of accuracy, the series $1, r^{\frac{1}{n}}, r^{\frac{2}{n}}, r^{\frac{3}{n}}, \dots, r^1, r^{\frac{n+1}{n}}, r^{\frac{n+2}{n}}, \dots, r^2, r^{\frac{2n+1}{n}}, \&c. \&c.$ may be considered as giving a system of logarithms to all the natural numbers 1, 2, 3, 4, &c. — and if continued backwards by negative indices, as giving the logarithms of all the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \&c.$ For example, if $r^{\frac{m}{n}} = 3$, $r^{-\frac{m}{n}}$ will be $\frac{1}{3}$, and $\frac{m}{n}$ will be the logarithm of 3, and $-\frac{m}{n}$ of $\frac{1}{3}$.

217. The logarithm of any particular number will depend * on the magnitude of r . — If r

* The logarithm of a given number, c , varies inversely as the magnitude of the ratio $r : 1$. For if $c = r^x$, and consequently x be the logarithm of c , $c : 1 = r^x : 1, = x \times r : 1$, and $x = \frac{c : 1}{r : 1} \propto$ (if c be given) $\frac{1}{r : 1}$.

= 10, the system agrees with the logarithms of the tables.

218. This quantity r , whose logarithm is unity may be called the *Basis* of the system, and the logarithm of a number may be defined to be the index of that power of the basis, which is equal to the proposed number. For if $r^x = c$, x is the logarithm of c .

219. The properties of logarithms may easily be deduced from this definition. Thus if $c = r^x$, and $d = r^y$, $cd = r^{x+y}$, and $x+y$ is the logarithm of cd ; or the sum of the logarithms of the factors is the logarithm of the product.

A table of common logarithms shews what power of 10 any number is. Thus because we find in the tables .9542425 the log. of 9, we infer that 9 is found by computation to be equal to $10^{.9542425}$ or 9 is the 9542425^{th} term nearly in the geometrical series 10.0000001 , 10.0000002 , 10.0000003 , &c.

Of the making of Logarithms.

220. A very small ratio is nearly doubled by doubling the difference between its terms,—tripled by tripling the difference,—and so on.

For if x be very small, the double of $1+x$: 1, viz. $(1+x)^2$: 1^2 , or $1+2x+x^2$: 1, is, on account of the

the exceeding smallness of x^2 in comparison with $2x$, nearly the same as $1 + 2x : 1$;—*In general*, the n multiple of $1 + x : 1$, viz. $1 + nx + n \cdot \frac{n-1}{2}$

$x^2 + \&c. : 1$, is, on account of the exceeding smallness of the higher powers of x in comparison with nx , nearly the same as $1 + nx : 1$.

221. Conversely a very small ratio is nearly halved, trisected, &c. by halving, trisecting, &c. the difference between the terms.

222. Cor. Very small ratios with a common consequent vary nearly as the excess of their antecedents above the consequent.

Hence if the logarithm of *one* very small ratio be obtained, the logarithm of *any other* very small ratio may be had.

223. Hence the logarithm of any number, c , may be found as follows:

Extract the square root of 10, and the square root of that square root, and so on. Let $1 + p$ denote the m^{th} root of 10 found in this manner, m being a large number and consequently p a very small fraction; then since $\log: 10 = 1$, the log. of $10^{\frac{1}{m}}$ or of $1 + p$ is $\frac{1}{m}$. Let $1 + q$ be the n^{th} root of

c , found in a similar manner, n being a large number; then (art. 222) $\text{Log. } 1 + q : \text{Log. } 1 + p :: p : q$
 $\therefore, \log.$

$$\therefore \log. 1+q \doteq \frac{p}{q} \times \log. 1+p = \frac{p}{q} \times \frac{1}{m};$$

$$\text{but since } c = \overline{1+q}^n, \log. c = n \times \log. \overline{1+q}$$

$$\therefore \log. c = \frac{np}{mq}.$$

Another method.

Find what two terms in the series 1, 10, 10², 10³, &c. the proposed number lies between.

Find a mean proportional between these two terms.

Between the mean thus found, and that extreme which differs least from the proposed number, find another mean proportional.

Between this second mean and the term which differs least from the proposed number, whether that term be the former mean or one of the original extremes, find another mean proportional and so on.

Then each successive mean will differ less and less from the proposed number, and at length a mean will be found, the difference between which and the proposed number will be within certain required limits.

For each geometrical mean between two terms find an arithmetical mean between the logarithms

or

or indices of those terms. Then each arithmetical mean will be the logarithm of the corresponding geometrical mean,—so that the last arithmetical mean will be the logarithm of the last geometrical mean, or of the proposed number nearly.

By these and similar laborious methods were the logarithms of almost all the prime * numbers in the tables originally computed;—various artifices were afterwards employed to lessen the labor, which the reader will find amply detailed in Dr. Hutton's very valuable Introduction to his tables:—at length the invention of infinite series superseded all other methods, and reduced the labor comparatively to nothing.

224. Small ratios with a common consequent being nearly proportional to the excesses of their antecedents above the consequent, it follows that in any system of logarithms the logarithms of numbers, which exceed unity by very small excesses, are either these excesses, or the same part, multiple or parts of them respectively. Thus if x and y be very small, and x be the logarithm of $1+x$, then in the same system, y is the logarithm of $1+y$;—if $5x$ be the logarithm of $1+x$, $5y$ is the logarithm of $1+y$;—if $\frac{x}{2}$ be the logarithm

* The logarithm of a composite number is found by adding together the logarithms of its factors.

of $1+x$, $\frac{y}{2}$ is the logarithm of $1+y$; &c.

225. Hence assuming f any very small fraction, and M the proper multiplier to produce the system we wish, Mf is the logarithm of $1+f : 1$, $2 \times Mf$, of $\sqrt{1+f}^2 : 1$ (or twice $\sqrt{1+f} : 1$); $3 \times Mf$, of $\sqrt[3]{1+f}^3 : 1$; &c. — Mf , of $\sqrt{1+f}^{-1} : 1$; — $2 \times Mf$ of $\sqrt{1+f}^{-2} : 1$; &c. &c. i. e. in the geometrical series continued both ways ad libitum

$$\sqrt[10]{1+f}, \sqrt[9]{1+f}, \sqrt[8]{1+f}, \sqrt[7]{1+f}, \sqrt[6]{1+f}, \sqrt[5]{1+f}, \sqrt[4]{1+f}, \sqrt[3]{1+f}, \sqrt[2]{1+f}, 1, \sqrt[2]{1+f}, \sqrt[3]{1+f}, \sqrt[4]{1+f}, \sqrt[5]{1+f}, \sqrt[6]{1+f}, \sqrt[7]{1+f}, \sqrt[8]{1+f}, \sqrt[9]{1+f}, \sqrt[10]{1+f}, \text{ \&c.}$$

$$\frac{1}{\sqrt[10]{1+f}}, \frac{1}{\sqrt[9]{1+f}}, \frac{1}{\sqrt[8]{1+f}}, \frac{1}{\sqrt[7]{1+f}}, \frac{1}{\sqrt[6]{1+f}}, \frac{1}{\sqrt[5]{1+f}}, \frac{1}{\sqrt[4]{1+f}}, \frac{1}{\sqrt[3]{1+f}}, \frac{1}{\sqrt[2]{1+f}}, \frac{1}{1}, \frac{1}{\sqrt[2]{1+f}}, \frac{1}{\sqrt[3]{1+f}}, \frac{1}{\sqrt[4]{1+f}}, \frac{1}{\sqrt[5]{1+f}}, \frac{1}{\sqrt[6]{1+f}}, \frac{1}{\sqrt[7]{1+f}}, \frac{1}{\sqrt[8]{1+f}}, \frac{1}{\sqrt[9]{1+f}}, \frac{1}{\sqrt[10]{1+f}}, \text{ \&c.}$$

the logarithm of any term is $Mf \times$ the index of that term; and if f be taken of proper minuteness, some one or other of the terms will coincide with any proposed number nearly. Thus if $f = .000\ 0001$, the 23025850^{th} term of the series is nearly equal to 10; for $10 = \sqrt[23025850]{1.0000001}$ nearly.

226. The less f is taken, the further must we proceed into the series before we find a term equal to a given number N . Moreover the index of the term that is equal to N , or the number denoting the distance of that term from the beginning of the series, varies inversely as f , nearly.

For in whatever proportion f is changed, in the same proportion nearly is changed the ratio $1+f : 1$; $\therefore$ the less f is taken, the greater must be the number of ratios $1+f : 1$ to make up the ratio

to $N : 1$. Or if $N : 1 = \overline{1+f}^n : 1 = n \times \overline{1+f} : 1$, $n \times \overline{1+f} : 1$ is constant, and n varies inverfely as $1+f : 1$, varies inverfely as f .

Hence $Mf \times n$, the * logarithm of N , remains the fame *nearly*, whatever changes take place in f , provided f be taken very fmall, and M remain constant; but if M varies, it varies in the fame ratio.

227. This magnitude M is called the *Modulus* of the fystem. If $M = 1$, the logarithms are called † Hyperbolic;—If $M = .43429448$ &c. they are the common logarithms; for ‡ this value of M will give 1 for the logarithm of 10.

A a 2

228.

* The accurate value of $\log. N$ is the limit to which $Mf \times n$ approaches, when f is diminished fine limite.

† Because the mixtilinear area, intercepted between any two ordinates to the afymtote of an hyperbola, is proportional to the ratio of the ordinates to each other; and the numerical values of thefe areas agree with the logarithms in the above fystem when $M = 1$, provided the parallelogram of the hyperbola be affumed as unity; an affumption, which though not at all neceffary, yet did naturally prefent itfelf in preference to any other to the writers who firft confidered the logarithmic properties of this curve.

‡ Allowing that the 23025850th term of the geometrical series $\overline{1.000,0001}^1, \overline{1.000,0001}^2$, &c. is equal to 10, it follows that $\log. 10 = M \times \{.0000001 \times 23025850$ (fee art. 225) and $M = \frac{\log. 10}{2.3025850} = (\text{if } \log. 10 = 1) \frac{1}{2.3025850} = .43429$ &c.

The

228. If x be a very small increment of x , the excess of the ratio $1+x+x' : 1$ above $1+x : 1$ viz. the ratio $1+x+x : 1+x$ or $1 + \frac{x}{1+x}$

: 1, varies nearly as $\frac{x}{1+x}$ (art. 222) and its loga-

rithm is $\frac{Mx}{1+x}$ nearly; $\therefore x$: the logarithm of

$1+x+x : 1+x :: 1+x : M$ nearly, or M the modulus of the system is the limiting value of a fourth proportional to the difference (x) between the terms of an indefinitely small ratio, its logarithm, and one of the terms.

The method of ascertaining what term of the series is equal to 10 may be as follows.

Let x be the index of that term; then $M \times .0000001 \times x = \log. 10$, and $x = \frac{\log. 10}{M \times .0000001}$. Now if $1+p$ be the m^{th} root of 10, (as in art. 223) then for the same reason that Mf is the $\log^m$. of $1+f$, Mp will be the $\log^m$. of $1+p$; hence $Mp = \log. 10^{\frac{1}{m}}$ and

$$mMp = \log. 10, \text{ and } \frac{mp}{.0000001} = \frac{\log. 10}{M \times .0000001} = x.$$

Without considering what term of the series 10 is, we have from the equation $mMp = \log. 10$, $M = \frac{\log. 10}{mp}$. And in a similar manner may the Modulus be found in any system, that is determined by assuming a logarithm for a particular ratio.

229. The value of the term $\overline{1+f}^{\frac{1}{f}}$ in the above series remains *nearly* the same, whatever changes be made in the small fraction f .*

$$\text{For } (\overline{1+f})^{\frac{1}{f}} : 1 = \frac{1}{f} \times (1+f : 1) \propto \frac{1}{f} \times f$$

nearly or is nearly constant,

230. The limiting value of $\overline{1+f}^{\frac{1}{f}}$ is the limit of the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c.$ *ad infinitum*.

For, if the numerator of f be *unity*, $\overline{1+f}^{\frac{1}{f}}$
 $= 1 + \frac{1}{f} \times f + \frac{1}{f} \cdot \frac{1-f}{2f} \times \frac{1}{f^2} + \&c. \text{ to } \frac{1}{f} + 1$
 terms *accurately*; and if we substitute $Y, X, W,$
 $\dots\dots Z$ for the compound co-efficients of $f, f^2,$
 $f^3, f^{\frac{1}{f}-1}$, respectively, this series becomes $1 + 1 +$
 $\frac{1}{2} + \frac{1}{2.3} \dots\dots\dots + \frac{1}{2.3.4 \dots \frac{1}{f}} + Yf + Xf^2 +$

* The fraction f may be taken so small that a certain number of places in the decimal value of $\overline{1+f}^{\frac{1}{f}}$ shall have their figures accurate, i. e. not liable to be changed by taking f still smaller; in other words, $\overline{1+f}^{\frac{1}{f}}$ has a limit, to which it approaches when f is diminished sine limite.

Wf

$Wf^3 \dots + Zf^i - 1$ accurately. Now by decreasing f , this latter part, which involves the powers of f , may (although the number of terms in it be thereby increased) be decreased sine limite; $\therefore$ the limit to which the whole approaches, when f is decreased sine limite, is the former part, $1 + 1 + \frac{1}{2} + \&c. \text{ ad infinitum.}$

231. The logarithm of $\overline{1+f}^{\frac{1}{f}} : 1$, viz. $Mf \times \frac{1}{f}$ is the modulus of the system. Hence the modulus is in all systems the measure of the same ratio. This ratio is found, by adding together the terms * of the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \&c.$ to be 2.718281828459 &c. : 1, and is called the *Modular Ratio*.

232. That the *same ratio* has the different moduli for its logarithms in the different systems appears also thus.

The measure of a *given* ratio varies as the modulus of the system; and the measure of a ratio in a *given* system varies as the magnitude of the ratio; $\therefore$ the measure of a ratio varies as the magnitude of the ratio and the Modulus jointly:

* The sum of 12 terms is accurate, as far as 6 places of decimals.

Hence

Hence the measure of the modular ratio varies as the magnitude of that ratio $\times$ the modulus; or $M \propto$ modular ratio $\times M$, and modular ratio is constant.

233. The number 2.718281 &c. is the basis of the system of hyperbolic logarithms. (see art. 218) For M , its logarithm, is in this system, unity.

234. Hence if $N = 2.718281 \&c.^x$, x is the hyperbolic logarithm of N .

Of the Logarithmic Series.

235. Let $1+a$ be the basis of the system; (see art. 218) and let $\sqrt[mx+nx^2+px^3+qx^4+\&c.]{} = 1+x$; then $mx+nx^2+\&c.$ is the logarithm of $1+x$.

Now this power expanded by the binomial Theorem, and arranged according to the powers of x ,

$$1 + ma - \frac{ma^2}{2} + \frac{ma^3}{3} - \frac{ma}{4} + \&c. \times x$$

* The meaning of this equation is, that the values of m , &c. are to be such, that if $1+a$ be raised to the power $mx+nx^2+\&c.$ and the result disposed according to the powers of x , the first term will be unity,—the simple power of x will have a compound co-efficient equal to unity,—and the higher powers of x will have compound co-efficients each equal to 0.

† Omitting the higher powers of x as not affecting the co-efficient of x , $\sqrt[mx+\&c.]{} = 1+mx + mx \times \frac{mx-1}{2} a^2 + mx \times \frac{mx-1}{2}$

$x + \&c.$ which $\therefore$ equals $1+x$; hence equating the co-efficients of the like powers of x on each

side, $1 = 1; m \times a - \frac{a^2}{2} + \frac{a^3}{3} - \&c. = 1$, and

$m = 1 \div a - \frac{a^2}{2} + \frac{a^3}{3} - \&c.$; and thus may the

succeeding co-efficients, n , p , &c. be found; but the law of their dependance on m would not appear by this method. In order to determine that,

assume as above; then $\log. 1 + \overline{x+z} = \overline{m.x+z} + \overline{n.x+z^2} + \overline{p.x+z^3} + \overline{q.x+z^4} + \&c. =$ (omitting to set down any but the first power of z , as the others will not be wanted) $mz + nx^2 + px^3 + \&c. + mz + 2nxz + 3p x^2 z + 4q x^3 z + \&c.$; hence $\log. 1 + \overline{x+z} - \log. 1 + \overline{x} = \overline{mz + 2nxz + 3p x^2 z + \&c.}$; but $\log. 1 + \overline{x+z} - \log. 1 + \overline{x} = \log. \frac{1 + \overline{x+z}}{1 + \overline{x}} =$ (by actual division) $(\log. 1 + \overline{z - xz + x^2 z - x^3 z + \&c.}) =$

$\overline{m.z - xz + x^2 z - x^3 z + \&c.} + \overline{n.z - xz + \&c.} + \overline{p \times \&c.} + \&c. =$ (omitting to set down the higher powers of z) $\overline{mz - mxz + mx^2 z - \&c.}$ Hence these two values of $\log. (1 + \overline{x+z}) - \log. (1 + \overline{x})$ are equal, or

$$\frac{mn-2}{3} a^3 + \&c. = (\text{omitting again the higher powers of } x)$$

$$1 + mxa - \frac{mxa^2}{2} + \frac{mxa^3}{3} - \&c.$$

mz

$$mx + 2nx^2 + 3px^3 + 4qx^4 + 5rx^5 + \&c. =$$

$$mx - mx^2 + mx^3 - mx^4 + mx^5 - \&c;$$

$\therefore$ equating the co-efficients of the like powers of x and x we have $m = m$;

$$2n = -m, \text{ and } n = -\frac{m}{2};$$

$$3p = m, \text{ and } p = \frac{m}{3}$$

$$4q = -m, \text{ and } q = -\frac{m}{4} \cdot \&c;$$

$$\therefore \log. 1+x = m \cdot x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.*$$

236. The first co-efficient, m , is the Modulus (M) of the system.

* Aliter. Let v be indefinitely small; then $(1+x)^v (= 1+vx + v \cdot \frac{v-1}{2} x^2 + \&c)$ approaches to a ratio of equality with

$$1+v \cdot x - \frac{x^2}{2} + \frac{x^3}{3} - \&c. \text{ on account of the exceeding smallness}$$

of the higher powers of v ; but because $v \cdot x - \frac{x^2}{2} + \&c.$ decreases

fine limite as v decreases, $\log. (1+v \cdot x - \frac{x^2}{2} + \&c.)$ approaches to

a ratio of equality with $M \times v \cdot x - \frac{x^2}{2} + \&c.$ (art. 225) Hence

$$\log. 1+x \text{ (which } = \frac{1}{v} \times \log. (1+x)^v) = M \times x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

B b

For

For if x decreases *sine limite*, $\log. 1+x$ approaches to a ratio of equality with Mx ; but upon the same supposition, $m. x - \frac{x^2}{2} + \&c.$, the

logarithm of $1+x$, approaches to a ratio of equality with mx ; $\therefore Mx = mx$, and $M = m$.

$$237. \text{ If } x = a, \text{ we have } \overline{1+a}^{m. a - \frac{a^2}{2} + \&c.} = 1+a;$$

$$\therefore m. a - \frac{a^2}{2} + \&c. = 1, \text{ and } m = 1 \div a - \frac{a^2}{2} + \&c$$

as was found in art. 235.

Substituting $-x$ and its powers for x and its powers, we shall in the same manner find $\log. 1-x$

$$= m. -x - \frac{x^2}{2} - \frac{x^3}{3} - \&c.$$

$$238. \text{ Hence } \log. \frac{1}{1-x} (= -\log. 1-x) =$$

$$m. x + \frac{x^2}{2} + \frac{x^3}{3} + \&c.$$

$$239. \text{ Hence } \log. \overline{1+x} - \log. \overline{1-x} (= \log. \frac{1+x}{1-x}) =$$

$$m. \left\{ \begin{array}{l} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c. \\ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c. \end{array} \right\} = 2m. x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c$$

240. By adding together the terms of the series $m, (x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.)$ the sums are alternately too great, and too small——If x be equal to or less than unity, these excesses and defects decrease *fine limite*, as more and more terms are taken; but if x be greater than unity, they increase; and in that case the series is of no direct use for finding an approximation to the logarithm of $1+x$.

241. The value of x in $\frac{1}{1-x}$ and in $\frac{1+x}{1-x}$ is always less than unity, whatever number these expressions are put equal to; and the series for their logarithms, viz. $m, (x + \frac{x^2}{2} + \frac{x^3}{3} + \&c.)$ and $2m, (x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.)$ always converge to the true values.

242. The latter series converges very fast if $\frac{1+x}{1-x}$ be a small number.

EXAMPLES.

1st. Required the Hyperbolic logarithm (*h.L.*) of 2, Putting $\frac{1+x}{1-x} = 2$, we have $x = \frac{1}{2}$, and *h.h.* of

$$2 = 2 \times \left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \&c. \right)$$

$$\frac{1}{3} = .33333333 \&c.$$

$$\frac{1}{3 \cdot 3^3} = .01234567 \&c.$$

$$\frac{1}{5 \cdot 3^5} = .00082307 \&c.$$

$$\frac{1}{7 \cdot 3^7} = .00006532 \&c.$$

$$\frac{1}{9 \cdot 3^9} = .00000564 \&c.$$

$$\frac{1}{11 \cdot 3^{11}} = .00000031 \&c.$$

$$.34657354$$

2

$$\text{Answer. } .69314708$$

This answer is correct to 6 places of decimals.

It would require at least 100000 terms of the se-

$$\text{ries } x - \frac{x^3}{2} + \frac{x^3}{3} - \&c.,$$

to obtain the *h.l.* of 2 to

the same degree of accu-

racy; for if $1+x=2$,

$x=1$ and the series be-

$$\text{comes } 1 - \frac{1}{2} + \frac{1}{3} - \&c.$$

whose 100000th term

converted into a decimal

gives an unit in the 6th

place.

2d. Required the *h.l.* of 5.

This might be found directly by putting $\frac{1+x}{1-x}$

= 5; but more easily by means of the logarithm of 2, thus

$$5 = 2^2 \times \frac{5}{4}, \therefore h.l. 5 = h.l. 2^2 + h.l. \frac{5}{4} =$$

$2 \times h.l. 2 + h.l. \frac{5}{4}$; and putting $\frac{5}{4} = \frac{1+x}{1-x}$, $x =$

$$\frac{1}{9}, \therefore h.l. \frac{5}{4} = 2 \times \left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \&c. \right)$$

$$\frac{1}{9} = .1111111 \&c.$$

$$\frac{1}{3 \cdot 9^3} = .0004572 \&c.$$

$$\frac{1}{5 \cdot 9^5} = .0000033 \&c.$$

$$\begin{array}{r} .1115716 \\ 2 \end{array}$$

$$h.l. \frac{5}{4} = .223143.$$

$$h.l. 4 = 1.386294$$

$$h.l. 5 = 1.609437 \text{ Ans}^r$$

243. Hence $h.l.$ of 10 ($= h.l. 2 + h.l. 5$) = $.693147 + 1.609437 = 2.302584$, which is correct to 5 places of decimals, the correct logarithm to 6 places being 2.302585.

Since the *Moduli* in different systems vary as the logarithms of any given number, and the common logarithm of 10 is 1,

$2.302585 : 1 :: 1$, the modulus of the Hyperbolic logarithms : the modulus of the common logarithms, which

which is therefore equal to $\frac{1}{2.302584} = .43429 \text{ \&c,}$
(see art. 227.)

244. To find the number in terms of the logarithm and its powers.

Let $x = \frac{x^2}{2} + \frac{x^3}{3} - \text{\&c}$ (the logarithm of $1+x$) = y ; then $1+x = \overline{1+a}^y$, $1+a$ being the basis: let $\overline{1+a}^y = 1 + by + cy^2 + dy^3 + \text{\&c}$;
then since $\overline{1+a}^y = 1 + ya + y \cdot \frac{y-1}{2} \times a^2$

$$+ y \cdot \frac{y-1}{2} \cdot \frac{y-2}{3} \times a^3 + \text{\&c} =$$

$$1 + y \cdot a - \frac{a^2}{2} + \frac{a^3}{3} - \text{\&c} + y^2 \times \text{\&c} + \text{\&c}$$

$$\text{we have } b = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} \text{\&c} = \frac{1}{m}$$

(art. 235) and in the same manner might the succeeding co-efficients $c, d, \text{\&c}$ be found; but in order to discover the law they follow, assuming as above, we have $\overline{1+a}^{y+z} = 1 + b \cdot \overline{y+z} + c \cdot$

$\overline{y+z}^2 + d \cdot \overline{y+z}^3 + \text{\&c.} =$ (omitting to set down the higher powers of z) $1 + by + cy^2 + dy^3 + \text{\&c} +$
 $bx + 2cyz + 3dy^2z + \text{\&c.}$

But $\overline{1+a}^{y+z} = \overline{1+a}^y \times \overline{1+a}^z =$ according
to the same assumption) $1 + by + cy^2 + dy^3 + \text{\&c}$
 $\times 1 + bz + \text{\&c} =$ (omitting to set down the
higher

higher powers of z) $1 + by + cy^2 + dy^3 + \&c$
 $+ bz + b^2yz + bcy^2z + bdy^3z + \&c.$

Hence, equating the two values of $\overline{1+a}^{y+z}$, and
 omitting the terms common to both, we have

$$bx + 2cyz + 3dy^2z + 4cy^3z + 5fy^4z + \&c$$

$$= bx + b^2yz + bcy^2z + bdy^3z + bcy^4z + \&c$$

and equating the co-efficients of the like powers
 of y and z on each side,

$$b = b = \frac{1}{m}; 2c = b^2, \text{ and } c = \frac{b^2}{2} = \frac{1}{2 \cdot m^2}$$

$$3d = bc = \frac{b^3}{2}, \text{ and } d = \frac{b^3}{2 \cdot 3} = \frac{1}{2 \cdot 3 \cdot m^3};$$

$$4e = bd = \frac{b^4}{2 \cdot 3}, \text{ and } e = \frac{b^4}{2 \cdot 3 \cdot 4} = \frac{1}{2 \cdot 3 \cdot 4 \cdot m^4};$$

&c. &c.

$$\text{and } 1+x = \overline{1+a}^y = 1 + \frac{y}{m} + \frac{y^2}{2 \cdot m^2} + \frac{y^3}{2 \cdot 3 \cdot m^3} +$$

$$\frac{y^4}{2 \cdot 3 \cdot 4 \cdot m^4} + \&c.$$

245. Cor. Let $y = 1$; then $\overline{1+a}^1 = 1+a = 1 +$
 $\frac{1}{m} + \frac{1}{2 \cdot m^2} + \frac{1}{2 \cdot 3 \cdot m^3} + \&c.$ which is the series

for the basis $(1+a)$ in terms of the modulus and
 its powers.

246. Cor. 2. Let $y = m$; then $\overline{1+a}^m = 1 + \frac{m}{m} + \frac{m^2}{2.m^2} + \&c. = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c$; hence $\overline{1+a}^m$ is independent of m and a^* ; but $\overline{1+a}^m : 1$ is the modular ratio; which is $\therefore$ constant.

247. If $m = 1$, we have $1+a = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \&c. = 2.71828 \&c$, which is $\therefore$ the basis of the hyperbolic logarithms.

248. If we substitute $-y$ and its powers for $+y$ and its powers, we shall in a similar manner find $\overline{1+a}^{-y} = 1 - \frac{y}{m} + \frac{y^2}{2.m^2} - \frac{y^3}{2.3.m^3} - \&c$, the terms being alternately $+$ and $-$.

Hence $\overline{1+a}^{-m} = 1 - \frac{m}{m} + \frac{m^2}{2.m^2} + \&c. = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \&c. = .367879 \&c$; and

$\overline{1+a}^{-m} : 1$, (the inverse of $\overline{1+a}^m : 1$) the inverse of the modular ratio, is that of 1 to .367879 $\&c$; and the modular ratio, which is 2.71828 $\&c : 1$ is also $\frac{1}{.367879 \&c} : 1$.

* a and m mutually depend on each other—and the change in one so compensates for the change in the other, as to keep the value of $\overline{1+a}^m$ unchanged.

249. The ratios between commensurable quantities are for the most part incommensurable one with another. Thus of all the ratios $2 : 1$, $3 : 1$, $4 : 1$, $5 : 1$, &c. *ad infinitum*, none are commensurable with the ratio $10 : 1$ except the ratios $100 : 1$, $1000 : 1$, $10000 : 1$, &c.

For if two ratios have unity for their consequents, and be commensurable, one antecedent must be an exact root or power of a root, of the other. Thus if $a : 1$ and $b : 1$ be commensurable, and $a : 1$ be $\frac{m}{n}$ ths of $b : 1$, by the definitions that establish the relative magnitudes of ratios, $a : 1 = b^{\frac{m}{n}} : 1$, and $a = b^{\frac{m}{n}}$.

Now none of the natural numbers 2, 3, 4, 5, &c. are powers or roots, or powers of roots of 10, except 100, 1000, &c.

250. If two unequal ratios be commensurable with each other, and be in their lowest integral terms, then, 1st, one of them at least must have both its terms exact squares or cubes or higher integral powers; and 2^{dly}, the antecedent of the other must be either some root of the antecedent of the former, that can be really extracted, or else some power of such root; and at the same time its consequent be the same extractable root or power of such root of the consequent of the former.

C c

For

For let $a : b$ and $c : d$ be the lowest integral terms of two unequal commensurable ratios; $a : b$ being the greater ratio; and let $\overline{a : b} = \frac{m}{n} \times$

$\overline{c : d}$ (m and n being prime to each other, and m greater than n) $= c^{\frac{m}{n}} : d^{\frac{m}{n}}$; then $a^n : b^n :: c^m : d^m$; and because $\frac{a}{b}$ and $\frac{c}{d}$ are in their lowest

terms, so is $\frac{a^n}{b^n}$ and its equal $\frac{c^m}{d^m}$ (art. 94); $\therefore a^n$

$= c^m$, and $b^n = d^m$; i. e. a^n and b^n are m powers. But from this it follows that a and b themselves must be m powers.—For if a^n be an m power, then, since a^m and consequently a^{pm} (pm being any multiple of m) is an m power, it follows that if n be subtracted as often as possible from m , and the remainder be made the index of a , that power of a will be an m power*—Again, if this remainder be subtracted as often as possible from n , and the new remainder be made the index of a , that power of a will be an m power; and thus we may proceed, the remainders continually decreasing, till,

* If the product of two factors and one of the factors be both m powers, the other factor is an m power.

For if $a^{pm} \times a^r$ be an m power, or in other words $\sqrt[m]{a^{pm}} \times \sqrt[m]{a^r}$ be rational, then, since $\sqrt[m]{a^{pm}}$ is rational, $\sqrt[m]{a^r}$ is rational; i. e. the m^{th} root of a^r can be extracted, or a^r is an m power.

on

on account of m and n being prime to each other, we come to the remainder *unity*, and prove that a^1 is an inpower. For example, if a^1 be a 15 power, a^{12} is a 15 power; but $a^{12} \times a^3$ is a 15 power; $\therefore a^3$ is a 15 power but $a^3 \times a^1$ is (by hyp.) a 15 power; $\therefore a^1$ is a 15 power. —In the same manner it may be shewn that b is an m power— $\therefore$ the 1st part of the assertion is true; and since $c = a^{\frac{n}{m}}$ and $d = b^{\frac{n}{m}}$, and it has been proved that the m roots of a and b are extractable, one or other alternative of the 2^d. assertion is true, according as n is equal to or greater than unity.

If the two given ratios involve surds, such multiples of them must be taken, as will render all the terms rational.—Then, if the resulting ratios be commensurable, so are the given ratios; otherwise not.

EXAMPLES.

1st. The two ratios 2 : 3 and 3 : 4 are incommensurable with each other. For neither 2 and 3 nor 3 and 4 are both squares or cubes or higher powers;

2^d. The two ratios 9 : 4 and 4 : 3 are incommensurable with each other. For though 9 and 4 be squares, yet 4 and 3 are neither roots nor powers of roots of 9 and 4 respectively.

3^d 8 : 27 is commensurable with 4 : 9—For 8 and 27 are cubes, and 4 and 9 are the $\frac{2}{3}$ powers of 8 and 27 respectively.

4th. Required whether $\sqrt[5]{3} : \sqrt[4]{2}$ be commensurable with $\sqrt[2]{3} : \sqrt[8]{2^5}$.

Multiplying the former ratio by 20 and the latter by 8, we have the ratios $3^4 : 2^5$ and $3^4 : 2^5$, which are equal, and $\therefore$ commensurable. $\therefore$ the given ratios are commensurable.

251. Hence it is evident that, *in any system the logarithms of most of the numbers 2, 3, 4, &c. must be incommensurable with unity.*

For if r be the *basis* of the system, no number can have a logarithm commensurable with *unity*, the logarithm of r , except those, whose ratios to *unity* are commensurable with the ratio $r : 1$; now no more of the ratios $2 : 1$, $3 : 1$, &c. can be commensurable with $r : 1$ than are commensurable one with another; and it has been shewn that very few of them are commensurable.

Thus if $2 : 1$ be commensurable with $r : 1$, $3 : 1$ is not, because incommensurable with $2 : 1$.

152. Hence if any set of magnitudes be accurate measures of the relative magnitudes of the ratios $2 : 1$, $3 : 1$, $4 : 1$, &c. these magnitudes have in
general

general no exact numerical value one in terms of another; i. e. *Numerical logarithms can only be approximations to the measures of the ratios of the different numbers to unity.**

253.

* All possible ratios, beginning from that of equality, and proceeding through all degrees of magnitude, together with their respective logarithms, may be conceived to be generated by the motion of points in right lines, thus:

Let AB be a fixt line, and let P and p set off at the same time from

A	B	P	Q	C	D	Y
		b	p	c	d	y

β and b respectively, and more along the indefinite right lines BY , by with motions so regulated that bp may always be proportional to the magnitude of the ratio $AP : AB$; i. e. if C, c , and D, d , be contemporary positions of P and p , so that bc may be to bd as the ratio $AC : AB$ to the ratio $AD : AB$; then bp will always be the logarithm of the ratio $AP : AB$,

In order to simplify the consideration, let the motion of P be such, that the ratio $AP : AB$ may increase uniformly, i. e. by equal increments in any equal times: Then

1st. P will continually move faster and faster.

For if PQ be a very small increment of AP , described in a given time, the corresponding increment of the ratio, viz. $\frac{AQ}{AB}$

$= \frac{AP}{AB}$ or $\frac{AQ}{AP} : 1$ or $1 + \frac{PQ}{AP} : 1$ must be constant; but $1 +$

$\frac{PQ}{AP} : 1$ varies nearly as $\frac{PQ}{AP}$ (see art. 222). $\therefore \frac{PQ}{AP}$ must be nearly

constant, or PQ , the increment *dato tempore*, must vary nearly as AP , and consequently the motion of P be continually accelerated.

2^dly.

253. The above observations are not at all inconsistent with what was delivered in Vol. I concerning

2^{dly}. The velocity of p will be constant. For since the ratio receives equal increments in equal times, so must it's measure, bp .

3^{dly}. Whatever velocity be determined to P , p may move with any uniform velocity whatsoever. For so long as the velocity of p is but constant, bp will always be proportional to the magnitude of the ratio $AP : AB$, and satisfy the definition of a logarithm.

4^{thly}. Hence the number of systems of linear logarithms is unlimited; because the number of different velocities, with which p may move, is unlimited.

5^{thly}. If AB be called 1, and the velocity of p be such, that at first setting off from B and b the motions of P and p are nearly alike, so nearly, that the ratio of the first co-temporary increments described by P and p may, by diminishing the time *sine limite*, be made to approximate *sine limite* to a ratio of equality, the logarithms are called hyperbolic. (see art. 227)

In this case when $AP = 10AB$, bp will be between 23 and 24 tenths of AB (see art. 243)

6^{thly}. If the velocity of p be so diminished from the above, that when $AP = 10AB$, $bp = AB$, the numerical values of the logarithms coincide with those in the common tables.

The motions of P and p in the Hyperbolic system, cannot even at first setting off be accurately alike, however small the time be taken because that of P is variable, and that of p uniform. It conveys a general and popular notion of their relative motions, to say that their *initial velocities* are equal—but this is not mathematical language, until a definition be given of velocity in variable motion—which cannot be done to any useful purpose without introducing the idea of limiting ratios.—To speak of velocity as a term accurately

concerning the addition and subtraction of ratios
 —That the sum of the ratios $a : b$ and $c : d$ is the ratio $ac : bd$ was there proved (see art. 297) from considering $c : d$ as some certain multiple, part or parts of $a : b$; now if $c : d$ be incommensurable with $a : b$, this demonstration is not strictly applicable—but because however small a ratio be proposed, some fraction, $\frac{m}{n}$, may be found such

that the difference between $c : d$ and $\frac{m}{n}$ ths of $a : b$

shall a less ratio than that proposed ratio, it may easily be shewn by a *reductio ad absurdum* that no other ratio but $ac : bd$ can be the sum of the ratios $a : b$ and $c : d$; and consequently either that ratio is their sum, or they have no sum.

Moreover, if we *define* $ac : bd$ to be the sum of $a : b$ and $c : d$, no argument will be required; and since such a definition is consistent with the properties of ratios, and indeed one from which the

accurately understood, and then seek to prove that two velocities are in the limiting ratio of the co-temporary increments, is similar to considering proportion as a well known affection of magnitudes, (which it would be but for the existence of incommensurables) and then endeavouring to demonstrate the 5th definition of the 5th Book of Euclid. This unphilosophical method was fully exposed by Dr. Barrow in the case of ratios—but an obscure and undefined usage of words still maintains its ground in many branches of the mathematics, and particularly in the doctrine of fluxions.

definition

definition given in art. 283, to determine the relative magnitude of ratios, immediately follows as a consequence, (see art. 306) it is evident that nothing untrue or inconsistent can in any instance result from calling $ac : bd$ the sum of $a : b$ and $c : d$.

SCHOLIUM.

254. An equation of the form $a^x = b$, where x , the unknown quantity, is an index, is said to be solved by Logarithms. For from this equation we have $\log. a^x = \log. b$; but $\log. a^x = x \times \log. a$ (art. 211) $\therefore x \times \log. a = \log. b$, and $x = \frac{\log. a}{\log. b}$

In a scientific view nothing more is seen in the equation $x = \frac{\log. a}{\log. b}$ than in the original equation $a^x = b$; for the logarithm of a number admits of no definition that directly points out the arithmetical operation necessary to find it.—It must be defined either directly by indices, or else by ratios; the consideration of whose relative values involves the consideration of indices.—To know the relation between $\log. a$ and $\log. b$ is to know what power of a is equal to b .

The only advantage therefore in a scientific view is, that in the expression $x = \frac{\log. a}{\log. b}$ the unknown quantity appears separated from the known quantities

quantities, and its dependence on them (provided care be taken to have a precise and accurate idea of the relative magnitude of ratios) sufficiently perspicuous.

But since computations have been made and inserted in tables, which serve to shew what power any one number is of any other, in a practical view the equation $x = \frac{\log. a}{\log. b}$ is of great uti-

lity in shortening the labours of actual computation.—For instead of seeking the value of x in the equation $a^x = b$ by trial, or by those indirect methods, which are actually employed in computing logarithms, and which have been explained in this chapter, we avail ourselves of those computations which have been already made, and the result of which is recorded in the tables.

presented and the dependence of the results on the method used to obtain them is discussed. It is also pointed out that the results are not to be taken too literally, but that they are of great value in showing the general character of the phenomena.

It is then shown that the results are in good agreement with the results obtained in other localities, and that the results are of great value in showing the general character of the phenomena. The results are also compared with the results obtained in other localities, and it is shown that the results are in good agreement with the results obtained in other localities.

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APPENDIX.

OF IMPOSSIBLE QUANTITIES, AND THE USE OF THE
NEGATIVE SIGN.

255. **T**HE rule in art. 174 for multiplying together *radical quantities*, will, if applied to *impossible quantities*, lead to an inconsistency. For $+\sqrt{-a} \times +\sqrt{-b}$ is not $+\sqrt{-a \times -b}$ or $+\sqrt{ab}$, because if $a = b$, $\sqrt{-a} \times \sqrt{-a}$ is the square of $\sqrt{-a}$, and $\sqrt{-a}$ is that expression, whose square is $-a$; therefore $\sqrt{-a} \times \sqrt{-a}$ is $-\sqrt{a^2}$, and consequently $\sqrt{-a} \times \sqrt{-b}$ should be $-\sqrt{ab}$.

The reason of this exception seems to have been misunderstood.—In order to discover it, we are not to make any inquiry into the supposed abstract nature of negative and impossible quantities, nor argue for and against the rule by a *reductio ad absurdum* derived from the application of other algebraic rules; because those arguments which are true when applied to real magnitudes,

may have no meaning whatever when applied to imaginary quantities. The whole matter may be reduced to this—If we change real equations by the application of certain symbols, thereby producing equations of no direct meaning, and then by reverse operations produce real equations again, these last equations must necessarily be true; but there can be no possible method of proving (nor does it always happen) that an imaginary equation, introduced by the operation of one rule, may be reduced to a real *true* equation by the operation of another. Now impossible quantities are introduced by placing the symbol $\sqrt{}$ over negative quantities; therefore we cannot be sure of not introducing error, if we reduce them to possible quantities any other way than by squaring, i. e. by taking away the radical sign.

It must however be allowed, that if we apply the *same* rule, whatever it be, to both sides of an equation, no *error* will arise, only a certain inconsistency in the names of things. Thus if $-a = -b$, $\sqrt{-a} = \sqrt{-b}$, and if we multiply both sides into $\sqrt{-a}$, by the rule in art. 174, the result, viz. $a = \sqrt{ab}$, is not *untrue*; but it is inconsistent to take a for the product of $\sqrt{-a}$ multiplied into itself, when we define $\sqrt{-a}$ to be that quantity which multiplied into itself produces $-a$.

In

In order to avoid both error and inconsistency, an impossible quantity ought to be separated into two factors, one of which is $\sqrt{-1}$, and the other a possible quantity; thus $\sqrt{-a} = \sqrt{-1} \times \sqrt{a}$; then in multiplying two impossible quantities together, the impossible parts, $\sqrt{-1}$, are not to be multiplied together by the rule in art. 174, but by taking off the sign; thus $\sqrt{-a} \times \sqrt{-b} = \sqrt{-1} \times \sqrt{a} \times \sqrt{-1} \times \sqrt{b} = (\sqrt{-1})^2 \times \sqrt{ab} = -1 \times \sqrt{ab} = -\sqrt{ab}$.

These impossible quantities are found sometimes to expedite demonstrations, but in the opinion of some excellent mathematicians, the disgusting jargon and air of mystery they introduce into mathematics, together with the confusion of ideas to which they give rise, are by no means compensated by their utility.

Similar objections have been made against the unlimited use of *negative* quantities; and undoubtedly it has occasioned many absurd notions and imperfect demonstrations; especially in the application of Algebra to Geometry; in which a line taken in one direction is assumed positive, and lines in the contrary direction negative, without any general demonstration that no error can result from such assumptions. But this does not seem to be a sufficient reason for restricting the use of

of the negative sign to denote merely the subtraction of a less quantity from a greater; because such a restriction would occasion boundless trouble, would introduce tedious circumlocution, and would often render obscure and difficult what otherwise appears plain and easy.

It might seem superfluous to point out any particular examples to justify this assertion, seeing that *no writer, who treats at large of the various branches of the Analytics, has refused to avail himself in some degree of the symbols called negative quantities*: yet as accusations of absurdity have by some intelligent men been preferred not only against the explanation given of these symbols, but against the very use of them in any case whatsoever, it may not be unacceptable to the reader, who on the one hand is disgusted with unmeaning words and imperfect demonstrations, and on the other hand is not so blinded by the very name of reform, as to refuse to hear any justification of established customs, to have the nature and use of these negative quantities briefly stated.

There are two principal ways of employing negative quantities, besides that, which serves merely to denote the subtraction of a less magnitude from a greater.

1st. By employing a letter, where a magnitude is to be subtracted, to denote that magnitude with its sign — prefixt.

Thus

Thus in problems, where it is a question whether a quantity should enter into an equation by addition or by subtraction, a letter with the sign + is put for it, and the question is determined according as the value of that letter comes out at last to be positive or negative.

Another example of the same nature we have in investigating the terms of a series, where it is unknown whether they be all positive, or some positive and some negative—letters with the sign + are put for the co-efficients of the terms without any consideration whether the terms be positive or negative. See the logarithmic Series.

The only thing to be considered in these and similar cases is, whether the rules we employ be such as lead necessarily to just conclusions; and whether the method be convenient.—If nothing more than the necessary justness of the conclusions be established, we are at least cleared from the imputation of false reasoning.

2^d. By setting the negative sign before a quantity, that is not employed independently, for the sake of convenient notation—as to generalize a rule, to express a theorem concisely, and shew the law which the terms follow.

Thus, from the extension of a rule, we have negative indices to denote the reciprocals of powers; as a^{-2} to denote $\frac{1}{a^2}$.

Again,

Again, if the letter m be employed, agreeably to the first head, to denote a number with its sign prefixt, the Binomial theorem is expressed generally by the equation $(1+x)^m = 1 + mx + m \cdot \frac{m-1}{2}$

$x^2 + \&c.$ where, if m be a fraction, the factors in the co-efficients will sooner or later become negative; viz. when the whole number to be subtracted becomes greater than m . And it does not seem possible to express this theorem generally, perspicuously, and unambiguously without employing these negative factors.

Are we then to give up strict demonstrations and convenient methods of reasoning, because some have indulged in reveries about quantities less than nothing—others have seriously set themselves to *prove* that $a^0 = 1$, and others perhaps have endeavoured to force their imaginations to conceive the nature of a *negative involution*?—Because the readers of these mysteries have often had their ideas perverted, their habits of strict reasoning impaired, and their taste for accurate demonstration vitiated?—Surely not; it rather behoves us to shew what these things *really are*, to distinguish words from ideas, and to explain in what cases rules are to be *assumed* instead of *proved*. In one word, the Mathematician ought to have no further anxiety about *negative quantities*,

quantities, than to be able strictly to demonstrate that the conclusions he obtains by their means must necessarily be true:—then whether they are to be employed or not, becomes merely a question of expediency.

OF THE DEMONSTRATION OF THE BINOMIAL THEOREM.

256. The co-efficients of the Binomial Theorem may be obtained with much more simplicity than in the demonstration in art. 331, Vol. I, by employing $1+x+y$ instead of $1+x+x$; as follows.

Let $m, b, c, \&c.$ be the co-efficients of the 2^{d} 3^{d} 4^{th} $\&c.$ terms of the m^{th} power of any binomial; and $m-1, b, c, \&c.$ the co-efficients of the same terms of the $m-1^{\text{th}}$ power: see art. 331, Vol. I. then $\overline{1+x+y}^m = 1 + m \cdot \overline{x+y} + b \cdot \overline{x+y}^2 + \&c. =$ (omitting to set down the higher powers of y , they not being wanted in this demonstration) $1 + mx + bx^2 + \&c. + my + 2bxy + 3cx^2y + 4dx^3y + \&c.$ and $\overline{1+x+y}^m = \overline{1+x}^m + m \cdot \overline{1+x}^{m-1} \times y + \&c. = 1 + mx + bx^2 + \&c. + my + m \cdot \overline{m-1} \cdot xy + mbx^2y + mcx^3y + \&c.$ hence, equating these two series, and omitting the common terms,

$$my + 2bxy + 3cx^2y + 4dx^3y + \&c. =$$

$my + m \cdot \overline{m-1} \cdot xy + mbx^2y + mcx^3y + \&c;$
 and equating the co-efficients of the like powers
 of x and y , $m = m$; $2b = \overline{m \cdot m-1}$, and $b =$
 $m \cdot \frac{m-1}{2}$; hence $b = \overline{m-1} \cdot \frac{m-2}{2}$. $3c = mb =$
 $\frac{\overline{m \cdot m-1} \cdot \overline{m-2}}{2}$, and $c = \frac{\overline{m \cdot m-1} \cdot \overline{m-2}}{2 \cdot 3}$; and so on.

This simplification was suggested by a theorem in
 Monsr. La Grange's "*Theorie des Fonctions Analyti-
 ques*," published in 1797. As it in a great measure
 destroys the value, which the author might
 have attached to his own original demonstration, he
 no longer hesitates to declare his opinion, that all
 the other algebraic demonstrations of the Binomial
 Theorem, which he has met with, are imperfect;
 being either not general, or not clear and satis-
 factory in demonstrating the law of the co-effi-
 cients, or employing quantities that are supposed to
 vanish; in which last case the demonstration is
 properly fluxional, but under a very disadvanta-
 geous form.

FINIS.

Fig. 1.

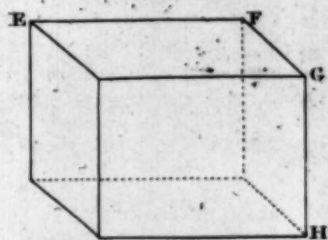
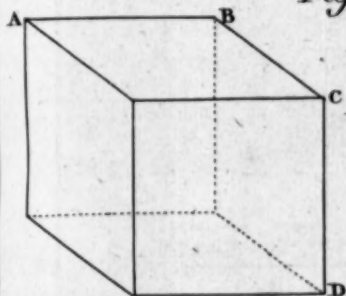


Fig. 2.

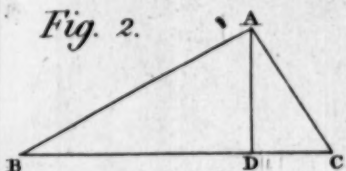


Fig. 3.

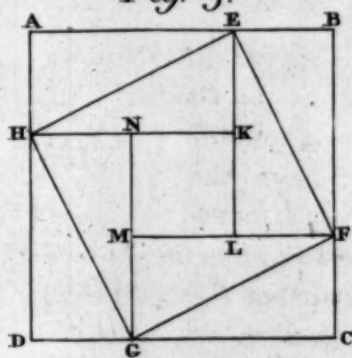


Fig. 4.

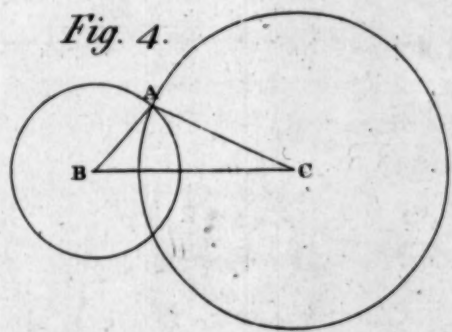


Fig. 5.

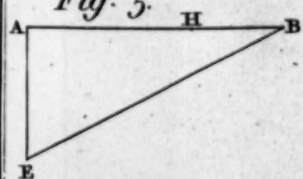


Fig. 6.

